Free Algebras and PI-Algebras

Graduate Course in Algebra

Vesselin Drensky

The book is devoted to the combinatorial theory of polynomial algebras, free associative and free Lie algebras, and algebras with polynomial identities. It also examines the structure of automorphism groups of free and relatively free algebras. It is based on graduate courses and short cycles of lectures presented by the author at several universities and its goal is to involve the reader as soon as possible in the research area, to make him or her able to read books and papers on the considered topics. It contains both classical and contemporary results and methods. A specific feature of the book is that it includes as its inseparable part more than 250 exercises and examples with detailed hints (50 % of the numbered statements), some of them treating serious mathematical results. The exposition is accessible for graduate and advanced undergraduate students with standard background on linear algebra and some elements of ring theory and group theory. The professional mathematician working in the field of algebra and other related topics also will find the book useful for his or her research and teaching.

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Drensky

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Author

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Vesselin Drensky
Institute of Mathematics and Informatics
Bulgarian Academy of Sciences
Sofia, Bulgaria

Preface

In September – December 1996 I had the pleasure to visit the Department of Mathematics of the University of Hong Kong. The present book contains an extended version of the graduate course on selected topics of algebra which I gave in the first semester of 1996/97. It is based on similar courses at the Department of Mathematics and Informatics of the University of Sofia and on short cycles of lectures presented at conferences and seminars.

The book is devoted to the combinatorial theory of polynomial algebras, free associative and free Lie algebras, and algebras with polynomial identities. It also examines the structure of automorphism groups of free and relatively free algebras.

The goal of the book is to involve the reader as soon as possible in the research area, to make him or her able to read books and papers on the considered topics and, hopefully, to commence research in some of the fields related to the book. Following this idea, I have included some classical results as well as some contemporary results and methods. Some results are given as exercises. Each chapter contains comments and references to the current situation in the discussed field which will be useful for the orientation of the reader.

I have tried to make the exposition accessible for graduate students with standard background on linear algebra and some elements of ring theory and group theory. Nevertheless I hope that a professional mathematician working in the field of algebra and other related topics also will find the book useful for his or her research.

Hong Kong – Sofia Vesselin Drensky

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I am very grateful to many of my colleagues and friends for the useful discussions on PI-algebras and suggestions for shortcuts of the proofs of some classical or new results. Without being exhaustive, I mention especially my advisers Georgy Genov from the University of Sofia and Yuri Bahturin from the Moscow State University, as well as S.A. Amitsur, L.L. Avramov, G.M. Bergman, L.A. Bokut, P.M. Cohn, M. Domokos, E. Formanek, M.B. Gavrilov, A. Giambruno, C.K. Gupta, N.D. Gupta, A.R. Kemer, P.E. Koshlukov, L.G. Makar-Limanov, G.M. Piacentini Cattaneo, A.P. Popov, Yu.P. Razmyslov, A. Regev, A.L. Shmelkin, I.B. Volichenko and E.I. Zelmanov. I am also thankful to my current and former Bulgarian students for the discussions which led to some improvements of proofs included in the present text.

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Introduction

This book is devoted to the theory of polynomial algebras, free associative and free Lie algebras, and algebras with polynomial identities. Presenting to the reader this area of mathematics, I shall try to answer two main questions:

I. *Why we study these topics?*

and

II. *How we study the objects?*

I Why We Study Free Algebras and PI-Algebras?

Our starting point is the assumption that the classes of all commutative and all finite dimensional algebras are important and have nice algebraic properties. It is natural to try to generalize their theory. Therefore, we have to find some reasonably big class of algebras which enjoys the most important properties of commutative and of finite dimensional algebras. We shall give a "naive" motivation on the following example. Let $R \subset \mathbb{C}^m$ be any set of points in the n-dimensional vector space.

Problem 1 (i) Find all algebraic equations $f(x_1, \ldots, x_m) = 0$ which vanish on R, i.e. all polynomials $f(x_1, \ldots, x_m) \in \mathbb{C}[x_1, \ldots, x_m]$ such that $f(r_1, \ldots, r_m) = 0$ for all $r_1, \ldots, r_m \in R$.

(ii) Find all solutions of the system of equations satisfied by R. Is this system of infinitely many equations equivalent to a finite system of equations?

(iii) Study the properties of the polynomial algebra modulo the ideal generated by these equations.

Problem 2 Describe the polynomials which vanish on "interesting" or "important" sets.

Problem 3 Describe the general properties of the set $R \subset \mathbb{C}^m$ defined as the set of solutions of some system of algebraic equations.

I think that there is no doubt that the attempt to solve the above problems was one of the main driving forces in the development of analytic geometry, linear algebra, commutative algebra and algebraic geometry.

Now, instead of $R \subset \mathbb{C}^m$ we consider an algebra R over a field K and try to translate the above problems in the language of the new situation.

Problem 1* (i) *Find all algebraic equations* $f(x_1, \ldots, x_m) = 0$ *which vanish on* R, i.e. $f(r_1, \ldots, r_m) = 0$ for all $r_1, \ldots, r_m \in R$. Answering the question what is an algebraic equation, leads us to the notions of a *free associative algebra* (an algebra of polynomials in noncommuting variables), a *polynomial identity*, a *PI-algebra* (an algebra with polynomial identity), and a *T-ideal* (an ideal which is invariant under all endomorphisms of the free algebra).

(ii) *Find all algebras satisfying the polynomial identities of* R. *Is this system of infinitely many identities equivalent to a finite system?* This problem gives rise to the notion of a *variety of algebras* and the *finite basis* or *Specht problem*: Is every T-ideal finitely generated as a T-ideal?

(iii) *Study the properties of the free algebra modulo the ideal generated by the polynomial identities of* R. This algebra is called a *relatively free algebra* and has many of the universal properties of the polynomial algebra.

Problem 2* Describe the polynomial identities which vanish on "interesting" or "important" algebras.

Problem 3* Describe the general properties of the algebra R, if the only information we have is that R satisfies *some* polynomial identity.

It turns out that the theory of PI-algebras is also related to other branches of mathematics as structure and combinatorial ring theory, the theory of finite dimensional division algebras, commutative and noncommutative invariant theory, projective geometry, etc.

II How We Study Free and PI-Algebras?

Reading a text book or a research article in mathematics we very often wonder how it was possible to discover such a complicated theorem. Every professional mathematician knows that studying some object one very rarely starts with technically complex considerations. Usually the investigation is based on some elementary reasoning and calculations in the easiest cases. In the moment when one has the proof in the simplest situations and the main ideas have been clarified enough, it is only a question of experience and a good mathematical background to state the result and its proof in general setup.

I have tried to show the theory of free algebras and algebras with polynomial identities from inside. I have included some classical results as well as given some contemporary results and methods. Our considerations are purely combinatorial and do not contain any structure theory of PI-algebras. This choice is based on my personal taste and on my believe that combinatorial ideas are easier to accept for a person not being involved before too deep in algebra. I think that for a graduate course it is not necessary to present the theorems in the most general form. Some of the proofs are given in important partial cases which illustrate the main idea. The proof of the general case is usually a matter of technique and I believe that the reader will be able, if necessary, to reconstruct the complete proofs. Some results are given as *exercises* to the main text. I prefer to reserve the word *problem* for an open problem. Trying to solve these exercises, one should first read the hints. If the hint finishes with *"see the paper or the book by ..."* this means that one faces big difficulties and one should consult the paper or the book. Some of the exercises contain serious results. I apologize to my colleagues and friends that I have treated some of their beautiful, important and complicated theorems in this way. I think that partial cases of these results included as exercises *do* show the kitchen of the considered topics. It is also a good idea, even if the reader has succeeded in solving an exercise without any assistance, to have a look at the original paper and to compare the solution with the original. I have added some comments and references for the current situation in the discussed fields. Without considering these remarks as a comprehensive survey, I hope that they will be useful for the orientation of the reader in the topics.

Main Topics

The first two chapters are introductory. They give the necessary background and fix the notation. Chapter 3 is devoted to the Specht problem and its negative solution for Lie algebras in characteristic 2. Chapter 4 deals with the reduction of arbitrary polynomial identities to polynomial identities of special form: homogeneous, multilinear and the so called proper polynomial identities and the relations between them. Chapter 5 contains illustrations on concrete examples: the polynomial identities of the Grassmann (or exterior) algebra and the algebra of upper triangular matrices as well as other algebras satisfying the same polynomial identities. Chapter 6 is devoted to commutative algebra and its applications to PI-algebras. I have also included as exercises some basic theorems of classical (or commutative) and noncommutative invariant theory. Chapter 7 discusses the polynomial identities for matrix algebras, the Amitsur-Levitzki theorem and central polynomials. The next two chapters consider the general properties of PI-algebras and show that, from combinatorial point of view, the PI-algebras are close to commutative and finite dimensional algebras. We discuss the theorem of Regev for the codimension sequence of the polynomial identities and the Shirshov theorem for finitely generated PI-algebras. The latter leads us to Gelfand-Kirillov dimension of finitely generated PI-algebras. Chapter 10 is devoted to the automorphisms of polynomial, free and relatively free algebras. Chapter 11 deals with free Lie algebras, their bases, subalgebras and automorphisms and with automorphisms of relatively free Lie algebras. Chapter 12 introduces the powerful method of representation theory of groups in the study of PI-algebras. I have also included the final test which I have given to my students from the University of Hong Kong and hints to the test.

Additional Readings

Here we give a short (and incomplete) list of books which can serve for further reading on some of the topics. Chapters of the books by Cohn [51] and Kharchenko [147] contain the theory of free associative algebras and their automorphisms and invariants. Concerning similar problems for free Lie algebras one can read the books by Bourbaki [37], Bahturin [21], Reutenauer [228] or Mikhalev and Zolotykh [184]. Good sources for the theory of algebras with polynomial identities are the books by Procesi [213], Jacobson [128] and Rowen [231]. For Lie algebras one should read chapters of the book by Bahturin [21]. The book by Hanna Neumann [192] gives the approach to free groups and groups with identical relations (which are the analogue of PI-algebras). I think that although devoted to group theory, this book is very useful also for ring theorists. Specific topics can be found in the literature listened in each section. We pay attention to the book by Krause and Lenagan [157] on Gelfand-Kirillov dimension and the book by Formanek [108] which is a good introduction to the polynomial identities and invariant theory of matrix algebras. Finally the want to mention the survey articles of Ufnarovski [253] and Bahturin and Olshanskii [23] which deal respectively with combinatorics of associative algebras and with parallel approach to algebraic systems with identical relations (including groups, associative and Lie algebras, etc.). As a very good starting point to the topics included in the present book as well as for many other topics in ring theory we recommend the two-volume book on ring theory by Rowen [232] or its one-volume student version [233].

1. Commutative, Associative and Lie Algebras

Throughout the book we fix the notation K for an arbitrary field of any characteristic. All considered vector spaces, algebras, modules, tensor products are over K. In order to unify the notation we use different fonts in the following way:
Greek: $\alpha, \beta, \ldots, \omega$ –"scalars" (elements of K) and mappings (usually homomorphisms);
Lower Case Italics: $a, b, \ldots, z$ – elements of vector spaces;
Upper Case Italics: $A, B, \ldots, Z$ – vector spaces;
Upper Case German: $\mathfrak{A}, \mathfrak{B}, \ldots, \mathfrak{Z}$ – classes of sets;
Upper Case Doubled: $\mathbb{C}, \mathbb{R}, \mathbb{Q}, \mathbb{Z}, \mathbb{N}$ – as usual, for the sets of complex, real, rational, integer and positive integer numbers.

In the first chapter we introduce the main objects to study: algebras (commutative, associative, Lie, etc.) over a field and free algebras, and give their basic properties. We prove Poincaré-Birkhoff-Witt Theorem (which describes the universal enveloping algebra of a Lie algebra) and some of its consequences which we use essentially in our exposition in the next chapters. We also introduce the Grassmann (or exterior) algebra which is one of the most important algebras in PI-theory.

1.1 Basic Properties of Algebras

Definition 1.1.1 A vector space R is called an *algebra* (or a *K-algebra*) if R is equipped with a binary operation $*$ (i.e. a mapping $* : (R, R) \to R$), called *multiplication*, such that for any $a, b, c \in R$ and any $\alpha \in K$

$$(a + b) * c = a * c + b * c,$$

$$a * (b + c) = a * b + a * c,$$

$$\alpha(a * b) = (\alpha a) * b = a * (\alpha b).$$

Usually we denote the multiplication of R by $\cdot$ (and write ab instead of $a \cdot b$), by $\times$, etc. Clearly, the notion of algebra generalizes both the notion of vector space and of ring. Initially we do not require $1 \in R$, the associativity of R, etc.

Remark 1.1.2 If the algebra R has a basis $\{e_i \mid i \in I\}$, then in order to define the multiplication in R it is sufficient to know the multiplication between the basis elements:

$$e_i * e_j = \sum_{k \in I} \alpha_{ij}^k e_k, \ \alpha_{ij}^k \in K,$$

where for fixed i and j only a finite number of α_{ij}^k are different from 0. Oppositely, for any given basis $\{e_i \mid i \in I\}$ of the vector space R and a given system of elements $\alpha_{ij}^k \in K$ with the property that for fixed i, j only a finite number of α_{ij}^k are not 0, we can define the multiplication in R by

$$\left(\sum_{i \in I} \xi_i e_i\right) * \left(\sum_{j \in I} \eta_j e_j\right) = \sum_{i,j \in I} \xi_i \eta_j (e_i * e_j), \ e_i * e_j = \sum_{k \in I} \alpha_{ij}^k e_k.$$

Definition 1.1.3 The subspace S of the algebra R is called a *subalgebra* if it is closed with respect to the multiplication, i.e. $s_1, s_2 \in S$ implies $s_1 * s_2 \in S$. The subalgebra I of R is called a *left ideal* of R if $RI \subseteq I$ (i.e. $r * i \in I$ for all $r \in R$, $i \in I$). Similarly one defines a *right ideal* and a *two-sided ideal* (or simply an *ideal*) (= left + right ideal in the same time, notation $I \triangleleft R$).

Exercise 1.1.4 Show that the following sets are K-algebras:

(i) L – any extension of the base field K with the ordinary operations;

(ii) $K[x]$, $K[x_1, \ldots, x_m]$ – the polynomials in one or several (commuting) variables; $K[x_1, x_2, \ldots]$ – the polynomial algebra in countably many variables;

(iii) $M_n(K)$ – the set of all $n \times n$ matrices with entries from K, with multiplication the usual multiplication of matrices; the set $\mathrm{End}_K(V)$ of all linear operators of a vector space V with the ordinary operations;

(iv) $U_n(K)$ – the subset of $M_n(K)$ consisting of all upper triangular matrices, with the usual multiplication;

(v) $sl_n(K)$ – the set of $n \times n$ matrices with trace zero and with multiplication

$$[r_1, r_2] = r_1 r_2 - r_2 r_1, \ r_1, r_2 \in sl_n(K);$$

$[r_1, r_2]$ is called the *commutator* of r_1 and r_2;

(vi) $S_n(K)$ – the set of all symmetric $n \times n$ matrices with entries from K and with multiplication

$$s_1 \circ s_2 = s_1 s_2 + s_2 s_1;$$

$O_n(K)$ – the set of all skew-symmetric $n \times n$ matrices with multiplication $[r_1, r_2]$.

(vii) The vector space KG with basis $\{g \mid g \in G\}$ where G is a finite group. The multiplication in KG is given by

$$\left(\sum_{g\in G}\alpha_g g\right)\left(\sum_{h\in G}\beta_h h\right) = \sum_{g,h\in G}\alpha_g\beta_h gh,\ \alpha_g,\beta_h\in K.$$

Here gh is the product of g and h in G. The algebra KG is called the *group algebra of the group G.*

Hint. (iv) Show that $U_n(K)$ is a subalgebra of $M_n(K)$.

(v) Use that $\mathrm{tr}(r_1r_2) = \mathrm{tr}(r_2r_1)$.

(vi) If r^t denotes the transpose of the matrix r, show that $s_1^t = s_1$, $s_2^t = s_2$ implies $(s_1\circ s_2)^t = s_1\circ s_2$, and $a_1^t = -a_1$, $a_2^t = -a_2$ gives $[a_1,a_2]^t = -[a_1,a_2]$.

Exercise 1.1.5 Let $\mathrm{char}K = 0$. Which of the algebras in Exercise 1.1.4 have nontrivial left ideals and which have nontrivial two-sided ideals?

Answer. The algebras in (ii), (iii) for $n > 1$ and $\dim V > 1$, (iv) for $n > 1$, (vii) for $|G| > 1$ possess nontrivial left ideals but in the case (iii) the algebra has trivial two-sided ideals only. In order to handle (vii), prove that $\{\alpha\sum_{g\in G} g \mid \alpha\in K\}$ is an ideal of KG.

Definition 1.1.6 The vector space homomorphism $\phi : R_1 \longrightarrow R_2$ of the algebras R_1, R_2 is an (algebra) *homomorphism* if

$$\phi(a*b) = \phi(a)*\phi(b),\ a,b\in R_1.$$

Similarly one introduces the notion of *isomorphism*, *automorphism* (= isomorphism of R and R), *endomorphism* (= homomorphism from R to R), etc.

The usual theorems concerning homomorphisms of vector spaces, groups and rings hold also for algebras. For example:

Theorem 1.1.7 *Let $\phi : R_1 \longrightarrow R_2$ be a homomorphism (of algebras). Then the kernel of ϕ*

$$\mathrm{Ker}(\phi) = \{r\in R_1 \mid \phi(r) = 0\}$$

is a two-sided ideal of R_1 and the factor algebra $R_1/\mathrm{Ker}(\phi)$ is isomorphic to the image $\mathrm{Im}(\phi) = \{\phi(r) \mid r\in R_1\}$ of ϕ.

Definition 1.1.8 Let R be an algebra over K.

(i) R is *associative* if $(a*b)*c = a*(b*c)$ for every $a,b,c\in R$;

(ii) R is *commutative* if $a*b = b*a$, $a,b\in R$;

(iii) R is a *Lie algebra* if for every $a,b,c\in R$,

$$a * a = 0, \quad \textit{the anticommutative law},$$

$$(a * b) * c + (b * c) * a + (c * a) * b = 0, \quad \textit{the Jacobi identity};$$

(iv) R is *unitary* (or *unital*) if R has a *unity* e with the property $e * r = r * e = r$, $r \in R$.

Very often we shall consider unitary algebras. In this case we make the convention that all subalgebras are also unitary with the same unity as the algebra. From this point of view, the nontrivial ideals of a unitary algebra are not subalgebras.

Exercise 1.1.9 (i) Show that the anticommutative law implies

$$a * b = -(b * a), \ a, b \in R.$$

(ii) If $\mathrm{char} K \neq 2$, show that the law

$$a * a = 0, \ a \in R,$$

is equivalent to the law

$$a * b = -(b * a), \ a, b \in R.$$

Hint. (i) Use that

$$0 = (a + b) * (a + b) = a * a + b * b + (a * b + b * a).$$

(ii) If $a * b = -(b * a)$, then for $a = b$ we obtain $2(a * a) = 0$.

Exercise 1.1.10 (i) Which of the algebras in Exercise 1.1.4 are associative, commutative, commutative-associative, unitary, Lie?

(ii) If R is an associative algebra (with multiplication $\cdot$), show that R is a Lie algebra with respect to the new multiplication $[r_1, r_2] = r_1 r_2 - r_2 r_1$, $r_1, r_2 \in R$. We denote this Lie algebra by $R^{(-)}$.

(iii) Show that the three-dimensional real vector space $\mathbb{R}^3$ is a Lie algebra with respect to the vector multiplication $\mathbf{a} \times \mathbf{b}$, $\mathbf{a}, \mathbf{b} \in \mathbb{R}^3$.

Exercise 1.1.11 Let R be an algebra and let the direct sum of vector spaces $R_1 = K \oplus R$ be equipped with multiplication

$$(\alpha_1 + r_1)(\alpha_2 + r_2) = \alpha_1\alpha_2 + (\alpha_1 r_2 + \alpha_2 r_1 + r_1 r_2), \ \alpha_1, \alpha_2 \in K, \ r_1, r_2 \in R.$$

Show that R_1 is a unitary algebra. (One says that R_1 is obtained from R by *formal adjoint of unity.*)

Convention 1.1.12 *If not explicitly stated, we always assume that the commutative associative algebras and the associative algebras are unitary.*

Exercise 1.1.13 Show that there exist commutative algebras which are not associative.

Definition 1.1.14 Let V and W be vector spaces with bases $\{v_i \mid i \in I\}$ and $\{w_j \mid j \in J\}$, respectively. The *tensor product* $V \otimes W = V \otimes_K W$ of V and W is the vector space with basis $\{v_i \otimes w_j \mid i \in I, j \in J\}$. We assume that

$$\left(\sum_{i\in I} \alpha_i v_i\right) \otimes \left(\sum_{j\in J} \beta_j w_j\right) = \sum_{i\in I}\sum_{j\in J} \alpha_i\beta_j (v_i \otimes w_j),\ \alpha_i, \beta_j \in K.$$

For the general definition of the tensor product of modules and its universal property see e.g. the book by Lang [161]. If V and W are algebras, then $V \otimes W$ is also an algebra with multiplication

$$(v' \otimes w')(v'' \otimes w'') = (v'v'') \otimes (w'w''),\ v', v'' \in V, w', w'' \in W.$$

Exercise 1.1.15 Show the isomorphism of algebras

$$K[x_1, \ldots, x_m] \cong K[x_1, \ldots, x_{m-1}] \otimes K[x_m].$$

Exercise 1.1.16 Show that the tensor product of associative algebras is also an associative algebra.

1.2 Free Algebras

Definition 1.2.1 Let $\mathfrak{V}$ be a class of algebras and let $F \in \mathfrak{V}$ be an algebra generated by a set X. The algebra F is called a *free algebra in the class* $\mathfrak{V}$*, freely generated by the set* X, if for any algebra $R \in \mathfrak{V}$, every mapping $X \to R$ can be extended to a homomorphism $F \longrightarrow R$. The cardinality $|X|$ of the set X is called the *rank* of F.

Example 1.2.2 For any set X the polynomial algebra $K[X]$ is free in the class of all unitary commutative associative algebras.

The proof of the following assertion is an easy exercise.

Proposition 1.2.3 *For every set* X *the algebra* $K\langle X\rangle$ *with basis the set of all words*

$$x_{i_1} \ldots x_{i_n},\ x_{i_j} \in X,\ n = 0, 1, 2, \ldots,$$

and multiplication defined by

$$(x_{i_1} \ldots x_{i_m})(x_{j_1} \ldots x_{j_n}) = x_{i_1} \ldots x_{i_m} x_{j_1} \ldots x_{j_n},\ x_{i_k}, x_{j_l} \in X,$$

is free in the class of all unitary associative algebras. If we consider the subspace of $K\langle X\rangle$ spanned by the words of length ≥ 1, we obtain the free nonunitary associative algebra, which is free in the class of all associative algebras.

Exercise 1.2.4 Let $K\{X\}$ be the vector space with basis the set of all nonassociative words, i.e. words of the form

$$(x_{i_1}\ldots)(\ldots x_{i_n}),\ x_{i_k}\in X,$$

where the parentheses are distributed in an arbitrary way. The multiplication in $K\{X\}$ is given by $u\cdot v=(u)(v)$ for any two words u,v. (More precisely, we omit the extra parentheses and, for example, write $x_i\cdot x_j=x_ix_j$ instead of $(x_i)(x_j)$, $x_i\cdot u=x_i(u)$ instead of $(x_i)(u)$, etc.) Show that this algebra is free in the class of all unitary algebras. It is called the *absolutely free algebra.* Since $K\langle X\rangle$ and $K\{X\}$ are generalizations of the polynomial algebra $K[X]$, we also call their elements polynomials (e.g. in noncommuting variables in the case of $K\langle X\rangle$).

Exercise 1.2.5 Prove that the rank of $K[X]$, $K\langle X\rangle$ and $K\{X\}$ is an invariant of the algebra, i.e. each of the isomorphisms $K[X]\cong K[Y]$, $K\langle X\rangle\cong K\langle Y\rangle$, $K\{X\}\cong K\{Y\}$ is equivalent to $|X|=|Y|$.

Hint. Let $\omega(K\langle X\rangle)$ be the ideal of $K\langle X\rangle$ generated by X, $\omega^0(K\langle X\rangle)=K\langle X\rangle$. Then

$$\dim\omega^k(K\langle X\rangle)/\omega^{k+1}K\langle X\rangle)=|X|^k,\ k=0,1,2,\ldots$$

Let $\phi:K\langle X\rangle\longrightarrow K\langle Y\rangle$ be an isomorphism such that $\phi(x_i)=f_i(y_1,\ldots,y_{n_i})$, $x_i\in X$. Then the composition $\psi=\phi\circ\theta$, where

$$\theta:K\langle X\rangle\longrightarrow K\langle X\rangle,\ \theta(x_i)=x_i-f_i(0,\ldots,0),\ x_i\in X,$$

is also an isomorphism and

$$\psi(\omega(K\langle X\rangle))=\omega(K\langle Y\rangle),$$

$$\dim\omega(K\langle X\rangle)/\omega^2(K\langle X\rangle)=\dim\omega(K\langle Y\rangle)/\omega^2(K\langle Y\rangle),\ |X|=|Y|.$$

1.3 The Poincaré-Birkhoff-Witt Theorem

Definition 1.3.1 If R is an associative algebra and the Lie algebra G is isomorphic to a subalgebra of $R^{(-)}$, we say that R is an *enveloping algebra* of G. The associative algebra $U = U(G)$ is the *universal enveloping algebra of the Lie algebra* G, if G is a subalgebra of $U^{(-)}$ and U has the following universal property: For any associative algebra R and any homomorphism of Lie algebras $\phi : G \longrightarrow R^{(-)}$ there exists a unique homomorphism of associative algebras $\psi : U \longrightarrow R$ which extends ϕ, i.e. ψ is equal to ϕ on G.

Poincaré-Birkhoff-Witt Theorem 1.3.2 *Every Lie algebra G possesses a unique (up to an isomorphism) universal enveloping algebra $U(G)$. If G has a basis $\{e_i \mid i \in I\}$, and the set of indices I is ordered, then $U(G)$ has a basis*

$$e_{i_1} \dots e_{i_p},\ i_1 \le \dots \le i_p,\ i_k \in I,\ p = 0, 1, 2, \dots .$$

Proof. Let the multiplication of G be given by

$$e_i \times e_j = \sum_{k \in I} \alpha_{ij}^k e_k,\ i, j \in I.$$

We consider the free associative algebra $K\langle X\rangle$, where $X = \{x_i \mid i \in I\}$ and its ideal J generated by all elements

$$[x_i, x_j] - \sum_{k \in I} \alpha_{ij}^k x_k,\ i, j \in I.$$

Let $U = K\langle X\rangle/J$ and let $y_i = x_i + J$, $i \in I$.

Proof of the Universal Property. Let $\iota : G \longrightarrow U^{(-)}$ be the vector space homomorphism defined by

$$\iota : \sum_{i \in I} \beta_i e_i \to \sum_{i \in I} \beta_i y_i,\ \beta_i \in K.$$

Clearly, ι is also a Lie algebra homomorphism because for any $e_i \times e_j$ (and by linearity for the product of any two elements of G),

$$\iota(e_i \times e_j) = \iota\left(\sum_{k \in I} \alpha_{ij}^k e_k\right) = \sum_{k \in I} \alpha_{ij}^k \iota(e_k) =$$

$$= \sum_{k \in I} \alpha_{ij}^k y_k = [y_i, y_j] = [\iota(e_i), \iota(e_j)].$$

Let R be an associative algebra and let $\phi : G \longrightarrow R^{(-)}$ be any Lie algebra homomorphism. Let $\phi(e_i) = r_i \in R$, $i \in I$. We define a homomorphism $\theta : K\langle X\rangle \longrightarrow R$ by $\theta(x_i) = r_i$, $i \in I$. Since

$$[r_i, r_j] = [\phi(e_i), \phi(e_j)] = \phi(e_i \times e_j) = \sum_{k\in I} \alpha_{ij}^k \phi(e_k) = \sum_{k\in I} \alpha_{ij}^k r_k,$$

we obtain that

$$[x_i, x_j] - \sum_{k\in I} \alpha_{ij}^k x_k \in \mathrm{Ker}\theta,\ J \subseteq \mathrm{Ker}\theta,$$

and we can define $\psi : K\langle X\rangle/J \longrightarrow R$ such that $\phi = \psi \circ \iota$. Prove the uniqueness for exercise!

Proof of the Embedding. We shall show that ι is an embedding of G into U. Let $y_{j_1} \dots y_{j_q}$ be any word (i.e. a monomial with coefficient 1) in U. If $j > i$, using the relations

$$y_j y_i = y_i y_j + \sum_{k\in I} \alpha_{ji}^k y_k,$$

we can express $y_{j_1} \dots y_{j_q}$ as a linear combination of words $y_{i_1} \dots y_{i_p}$ with $i_1 \leq \dots \leq i_p$ and $p \leq q$. It is sufficient to show that these elements $y_{i_1} \dots y_{i_p}$ are linearly independent in U.

In the vector space $K\langle Z\rangle$, $Z = \{z_i \mid i \in I\}$, we define linear operators called *reductions* in the following way. For a fixed word

$$u = az_j z_i b = z_{i_1} \dots z_{i_s} z_j z_i z_{j_1} \dots z_{j_t},\ j > i,$$

the reduction replaces u by

$$a(z_i z_j + \sum_{k\in I} \alpha_{ji}^k z_k)b,$$

and on all other words it acts identically. Clearly, for any $f \in K\langle Z\rangle$, there exists a finite sequence of reductions which brings f to a linear combination of words $z_{k_1} \dots z_{k_p}$, $k_1 \leq \dots \leq k_p$, and we call this linear combination the *reduced form of* f. The crucial moment of the proof is the following lemma.

Lemma 1.3.3 *The reduced form of $f \in K\langle Z\rangle$ is unique.*

Proof of Lemma 1.3.3. We order the monomials of $K\langle Z\rangle$ comparing them first by degree and then lexicographically:

$$\alpha z_{i_1} \dots z_{i_m} > \beta z_{j_1} \dots z_{j_n},\ 0 \neq \alpha, \beta \in K,$$

if either $m > n$ or $m = n$ and $i_1 = j_1, \dots, i_s = j_s$, $i_{s+1} > j_{s+1}$ for some $s < m$. This is the so called *deg-lex ordering* of the monomials in $K\langle Z\rangle$. Applying a reduction ρ to a monomial u, either $\rho(u) = u$ or the leading term of $\rho(u)$ is smaller than u. We apply induction on the deg-lex ordering, assuming that if the leading term of $g \in K\langle Z\rangle$ is smaller than the leading term of f, then g has a unique reduced form. The base of the induction is for f being linear when we cannot apply any nontrivial reductions to f and it coincides with its reduced form. The idea of the proof is to show that for any two reductions ρ and σ, there exist reductions $\rho_1, \dots, \rho_k$ and $\sigma_1, \dots, \sigma_l$ such that

$$(\rho_1 \circ \dots \circ \rho_k)(\rho(f)) = (\sigma_1 \circ \dots \circ \sigma_l)(\sigma(f))$$

and then to apply the inductive arguments. It is easy to see that the only difficulties are in the case

$$\rho(z_3 z_2 z_1) = \left(z_2 z_3 + \sum_{k \in I} \alpha^k_{32} z_k \right) z_1,$$

$$\sigma(z_3 z_2 z_1) = z_3 \left(z_1 z_2 + \sum_{l \in I} \alpha^l_{21} z_l \right).$$

We define the reductions as follows: ρ_1 acts on $z_2 z_3 z_1$ and changes the places of z_1, z_3:

$$\rho_1 : z_2 z_3 z_1 \to z_2 z_1 z_3 + \dots;$$

similarly

$$\rho_2 : z_2 z_1 z_3 \to z_1 z_2 z_3 + \dots,$$

$$\sigma_1 : z_3 z_1 z_2 \to z_1 z_3 z_2 + \dots,$$

$$\sigma_2 : z_1 z_3 z_2 \to z_1 z_2 z_3 + \dots.$$

Direct calculations give

$$(\rho_2 \circ \rho_1 \circ \rho)(z_3 z_2 z_1) = (\rho_2 \circ \rho_1)(z_2 z_3 z_1 + \sum_k \alpha^k_{32} z_k z_1) =$$

$$= \rho_2(z_2 z_1 z_3) + \sum_m \alpha^m_{31} z_2 z_m + \sum_k \alpha^k_{32} z_k z_1 =$$

$$= z_1 z_2 z_3 + \sum_l \alpha^l_{21} z_l z_3 + \sum_m \alpha^m_{31} z_2 z_m + \sum_k \alpha^k_{32} z_k z_1,$$

$$(\sigma_2 \circ \sigma_1 \circ \sigma)(z_3 z_2 z_1) = (\sigma_2 \circ \sigma_1)(z_3 z_1 z_2 + \sum_l \alpha^k_{21} z_3 z_l) =$$

$$= \sigma_2(z_1 z_3 z_2) + \sum_m \alpha^m_{31} z_m z_2 + \sum_l \alpha^l_{21} z_3 z_l =$$

$$= z_1 z_2 z_3 + \sum_l \alpha^k_{32} z_1 z_k + \sum_m \alpha^m_{31} z_m z_2 + \sum_l \alpha^l_{21} z_3 z_l.$$

Now we apply further reductions, if necessary, in order to present all $z_j z_i$, $j > i$, as $z_i z_j +$ linear terms. In this way $r = (\rho_2 \circ \rho_1 \circ \rho)(z_3 z_2 z_1)$ and $s = (\sigma_2 \circ \sigma_1 \circ \sigma)(z_3 z_2 z_1)$ are reduced to linear combinations of $z_1 z_2 z_3$, $z_i z_j$, z_i. The coefficient of $z_1 z_2 z_3$ in the reduced form of r and s is equal to 1, the coefficients of $z_i z_j$ in the reduced form of r and s are also the same. Finally, writing down the Jacobi identity for e_1, e_2, e_3 together with the anticommutativity ($\alpha^k_{ij} = -\alpha^k_{ji}$), we get some relations between the products $\alpha^k_{ij} \alpha^l_{pq}$ which give that the coefficients of z_i are also equal (check it!). In this way the reduced forms of r and s are the same and this completes the proof of the lemma.

Now we complete the proof of the embedding. We consider an algebra W with basis

$$\{z_{i_1} \dots z_{i_p} \mid i_1 \leq \dots \leq i_p, p \geq 0\},$$

and multiplication

$$(z_{i_1} \dots z_{i_p})(z_{j_1} \dots z_{j_q}) = \text{ the reduced form of } z_{i_1} \dots z_{i_p} z_{j_1} \dots z_{j_q}.$$

By Lemma 1.3.3, the multiplication is associative, i.e. W is an associative algebra. The kernel of the canonical homomorphism $\pi : K\langle X\rangle \to W$ (defined by $\pi(x_i) = z_i$) contains all $[x_i, x_j] - \sum_k \alpha_{ij}^k x_k$, i.e. $J \subseteq \mathrm{Ker}\pi$, and W is a homomorphic image of U. Since the images $z_{i_1} \dots z_{i_p}$ of $y_{i_1} \dots y_{i_p}$, $i_1 \leq \dots \leq i_p$, are linearly independent in W, the elements $y_{i_1} \dots y_{i_p}$ are also linearly independent and we obtain that $\mathrm{Ker}\pi = J$, i.e. $U \cong W$ and this completes the proof of the theorem.

Exercise 1.3.4 (i) If G and H are Lie algebras and H is a homomorphic image of G, then $U(H)$ is a homomorphic image of $U(G)$.

(ii) $U(G_1 \oplus G_2) \cong U(G_1) \otimes U(G_2)$.

(iii) The algebra $U(G)$ has no zero divisors.

Hint. (iii) See e.g. the book by Bourbaki [37]. See the same book for further properties of the universal enveloping algebras of Lie algebras.

Theorem 1.3.5 (Witt) *The Lie subalgebra $L(X)$ of $K\langle X\rangle^{(-)}$ generated by X is isomorphic to the free Lie algebra with X as a set of free generators; $U(L(X)) = K\langle X\rangle$.*

Proof. Let R be any associative algebra and let $\phi : L(X) \longrightarrow R^{(-)}$ be a homomorphism. The mapping $\phi_0 : X \to R$ defined by $\phi_0(x) = \phi(x)$, $x \in X$, induces a unique homomorphism $\psi : K\langle X\rangle \longrightarrow R$. Since $\phi(x) = \psi(x)$, we obtain that the restriction of ψ on $L(X)$ is equal to ϕ. Hence $U(L(X)) = K\langle X\rangle$. If G is a Lie algebra and R is its enveloping algebra, then any mapping $X \to G \subset R$ induces a homomorphism $K\langle X\rangle \longrightarrow R$; its restriction on $L(X)$ is a homomorphism of $L(X)$ to $R^{(-)}$ which sends the generators of $L(X)$ to G. Therefore the image of $L(X)$ is in G and this gives that $L(X)$ is free in the class of all Lie algebras.

Definition 1.3.6 Let M be a vector space, let R be an associative algebra and let

$$\rho : R \longrightarrow \mathrm{End}_K(M)$$

be an algebra homomorphism (such that $\rho(1) = \mathrm{id}$). Then ρ is called a *representation of R in M* and M is a *left R-module.* Similarly one defines a *right R-module* assuming that the linear operators of M act from the right.

Exercise 1.3.7 Show that every left ideal of R is a left R-module, where the left action of $r \in R$ is given by $\rho(r) : s \to rs$, $s \in R$.

Proposition 1.3.8 *Let H be a subalgebra of the Lie algebra G, and let H be generated by $h_1, \ldots, h_m$. Let the right $U(G)$-module $GU(G)$ be generated by $h_1, \ldots, h_m$. Then $G = H$.*

Proof. Let $H \neq G$. We consider an arbitrary basis $\{h_i \mid i \in I\}$ of H and extend it by a set $\{g_j \mid j \in J\} \subset G \setminus H$ to a basis of G, assuming that $h_i < g_j$. Since the right $U(G)$-module $GU(G)$ is generated by H, if $g \in G \setminus H$ is a basis element of G, then $g = g \cdot 1 \in GU(G)$ and

$$g = \sum_{k=1}^{m} h_k f_k,\ f_k \in U(G).$$

We express each f_k as in Poincaré-Birkhoff-Witt Theorem 1.3.2:

$$g = \sum_{k=1}^{m} h_k \sum_{p,q} \beta_{kpq} h_{p_1} \ldots h_{p_s} g_{q_1} \ldots g_{q_t},\ \beta_{kpq} \in K,$$

and bring g to its reduced form. Clearly, each product $h_k h_{p_1} \ldots h_{p_s}$ is a linear combination of products of h_i's, i.e. in the reduced form of g each summand contains some h_i. This contradicts with the linear independence of the reduced basis of $U(G)$. Hence $G = H$.

Remark 1.3.9 Using the same idea as in the proof of Poincaré-Birkhoff-Witt Theorem 1.3.2 one can develop Gröbner bases techniques both for commutative and noncommutative algebras. The Gröbner bases are a very powerful tool in computational algebra, algebraic geometry and invariant theory, see e.g. the books by Adams and Loustaunau [4] and Sturmfels [247]. For applications to noncommutative algebra we forward to the paper by Bergman [33], the surveys by Ufnarovski [253], Mora [186] and Belov, Borisenko and Latyshev [27] and the lecture notes by Latyshev [167].

Definition 1.3.10 Let V be a vector space with ordered basis $\{e_i \mid i \in I\}$.

(i) The *Grassmann* (or *exterior*) algebra $E(V)$ of V is the associative algebra generated by $\{e_i \mid i \in I\}$ and with *defining relations*

$$e_i e_j + e_j e_i = 0,\ i, j \in I,$$

(and $e_i^2 = 0$ if $\mathrm{char} K = 2$). This means that $E(V)$ is isomorphic to the factor algebra $K\langle X\rangle/J$, where $X = \{x_i \mid i \in I\}$ and the ideal J is generated by $x_i x_j + x_j x_i$, $i, j \in I$. If $\dim V$ is countable, we assume that V has a basis $\{e_1, e_2, \ldots\}$ and denote $E(V)$ by E.

(ii) If V is equipped with a symmetric bilinear form $\langle\, ,\, \rangle$, the *Clifford algebra* of V is generated by the basis of V with defining relations

$$e_i e_j + e_j e_i = \langle e_i, e_j \rangle,\ i, j \in I.$$

Exercise 1.3.11 In the notation of Definition 1.3.10, show that the Grassmann and Clifford algebras of the vector space V have (the same) basis

$$e_{i_1} \dots e_{i_n},\ i_1 < \dots < i_n,\ i_k \in I,\ n = 0, 1, 2, \dots .$$

Hint. Apply reductions as in the proof of Poincaré-Birkhoff-Witt Theorem 1.3.2.

2. Algebras with Polynomial Identities

A significant part of the book is devoted to algebras with polynomial identities. In this chapter we introduce PI-algebras and the related with them notions of varieties of algebras and relatively free algebras. We also give some examples of PI-algebras which will motivate our further study. Finally, we prove the theorem of Birkhoff which describes varieties of any algebraic systems in categorical language.

2.1 Definitions and Examples of PI-Algebras

From now on we fix a countable infinite set $X = \{x_1, x_2, \ldots\}$. Sometimes we shall also use other symbols, (e.g. y, z, y_i, z_j, etc.) for the elements of X.

Definition 2.1.1 (i) Let $f = f(x_1, \ldots, x_n) \in K\langle X\rangle$ and let R be an associative algebra. We say that $f = 0$ is a *polynomial identity for* R if

$$f(r_1, \ldots, r_n) = 0 \text{ for all } r_1, \ldots, r_n \in R.$$

Sometimes we shall also say that f itself is a polynomial identity for R.

(ii) If the associative algebra R satisfies a nontrivial polynomial identity $f = 0$ (i.e. f is a nonzero element of $K\langle X\rangle$), we call R a *PI-algebra* ("PI" = "Polynomial Identity").

Exercise 2.1.2 Show that $f \in K\langle X\rangle$ is a polynomial identity for R if and only if f is in the kernel of all homomorphisms $K\langle X\rangle \longrightarrow R$.

Examples 2.1.3 (i) The algebra R is commutative if and only if it satisfies the polynomial identity

$$[x_1, x_2] = x_1x_2 - x_2x_1 = 0.$$

(ii) Let R be a finite dimensional associative algebra and let $\dim R < n$. Then R satisfies the *standard identity of degree* n

$$s_n(x_1, \ldots, x_n) = \sum_{\sigma\in S_n} (\text{sign}\sigma) x_{\sigma(1)} \ldots x_{\sigma(n)} = 0,$$

where S_n is the symmetric group of degree n. The algebra R also satisfies the *Capelli identity*

$$d_n(x_1,\ldots,x_n;y_1,\ldots,y_{n+1}) = \sum_{\sigma\in S_n}(\text{sign}\sigma)y_1x_{\sigma(1)}y_2\ldots y_nx_{\sigma(n)}y_{n+1} = 0.$$

Hint. (ii) Since both the standard and the Capelli polynomials are skew-symmetric in $x_1,\ldots,x_n$, it is sufficient to verify that $s_n = 0$ and $d_n = 0$ on the elements of a fixed basis of R. Since $\dim R < n$, we obtain that s_n and d_n vanish on R.

Exercise 2.1.4 Show that the Grassmann algebra E satisfies the polynomial identity

$$[x_1,x_2,x_3] = [[x_1,x_2],x_3] = 0.$$

Hint. Since $[x_1,x_2,x_3]$ is linear in each of the variables x_1,x_2,x_3, it is sufficient to see that $[r_1,r_2,r_3] = 0$ for the basis elements of E. Check that

$$[r_1,r_2] = [e_{i_1}\ldots e_{i_m}, e_{j_1}\ldots e_{j_n}] = (1-(-1)^{mn})e_{i_1}\ldots e_{i_m}e_{j_1}\ldots e_{j_n},$$

i.e. $[r_1,r_2]\neq 0$ implies that both m and n are odd integers. Then $[r_1,r_2]$ is of even length.

Definition 2.1.5 The (*Lie*) *commutator of length* n, $n>1$, is defined inductively by

$$[x_1,x_2] = x_1(\text{ad}x_2) = x_1x_2 - x_2x_1,$$

$$[x_1,\ldots,x_{n-1},x_n] = [[x_1,\ldots,x_{n-1}],x_n],\ n>2.$$

Instead of the above *left normed* commutators, some books (e.g. Bourbaki [37] and Bahturin [21]) use the right normed notation

$$[x_1,x_2] = (\text{ad}x_1)x_2,\ [x_1,x_2,x_3] = [x_1,[x_2,x_3]].$$

Exercise 2.1.6 Show that the tensor product $E\otimes E$ satisfies the *centre-by-metabelian identity*

$$[[x_1,x_2],[x_3,x_4],x_5] = 0.$$

Exercise 2.1.7 Let $M_2(K)$ be the 2×2 matrix algebra. Show that $M_2(K)$ satisfies the following polynomial identities:

(i) The standard identity $s_4(x_1,x_2,x_3,x_4) = 0$.

(ii) The *Hall identity* $[[x_1,x_2]^2,x_3] = 0$.

(iii) Show that $M_2(K)$ does not satisfy the Capelli identity $d_4 = 0$ and the standard identity $s_3 = 0$.

Hint. (i) Fix the basis of $M_2(K)$

$$\{e = e_{11} + e_{22}, e_{11}, e_{12}, e_{21}\}$$

and rewrite the standard polynomial s_4 in the form

$$s_4 = [x_1, x_2] \circ [x_3, x_4] + [x_1, x_3] \circ [x_4, x_2] + [x_1, x_4] \circ [x_2, x_3].$$

Then $s_4(e, e_{11}, e_{12}, e_{21}) = 0$ because e participates in commutators only.

(ii) For any 2×2 matrix r the Cayley-Hamilton theorem gives that

$$r^2 - \mathrm{tr}(r)r + \det(r)e = 0.$$

If $r = [r_1, r_2]$, then $\mathrm{tr}(r) = 0$ and r^2 is a scalar matrix.

(iii) Find elements of $M_2(K)$ such that the Capelli polynomial d_4 does not vanish on them. Do the same for the standard polynomial s_3.

Exercise 2.1.8 Show that the $n \times n$ matrix algebra $M_n(K)$ satisfies the *identity of algebraicity*

$$d_{n+1}(1, x, x^2, \ldots, x^n; 1, y_1, \ldots, y_n, 1) =$$
$$= \sum_{\sigma \in S_{n+1}} (\mathrm{sign}\sigma) x^{\sigma(0)} y_1 x^{\sigma(1)} y_2 \ldots y_n x^{\sigma(n)} = 0,$$

where the symmetric group S_{n+1} acts on $\{0, 1, \ldots, n\}$, and the identity

$$s_n([x, y], [x^2, y], \ldots, [x^n, y]) = 0.$$

Hint. Use the Cayley-Hamilton theorem to conclude that $1, x, x^2, \ldots, x^n$ are linearly dependent. Do the same conclusion for $[x, y], [x^2, y], \ldots, [x^n, y]$.

Exercise 2.1.9 Let $U_n(K)$ be the algebra of $n \times n$ upper triangular matrices. Show that $U_n(K)$ satisfies the identity

$$[x_1, x_2] \ldots [x_{2n-1}, x_{2n}] = 0.$$

Derive from here that $U_n(K)$ satisfies also the standard identity of degree $2n$. (Later we shall see in Theorem 7.1.3, that $M_n(K)$ satisfies $s_{2n} = 0$ and this is the statement of the famous Amitsur-Levitzki theorem.)

Hint. Show that $[r_1, r_2]$ is an upper triangular matrix with zero diagonal, $r_1, r_2 \in U_n(K)$, and use that the product of n such zero diagonal matrices is 0. Then rewrite s_{2n} as

$$\frac{1}{2^n} \sum_{\sigma \in S_{2n}} (\mathrm{sign}\sigma) [x_{\sigma(1)}, x_{\sigma(2)}] \ldots [x_{\sigma(2n-1)}, x_{\sigma(2n)}].$$

Exercise 2.1.10 (i) Show that if R is a finite dimensional algebra over a finite field K, then R satisfies a nonzero polynomial identity in one variable.

(ii) If the base field K is finite and $|K| = q$, show that there exists a positive integer m such that $M_n(K)$ satisfies the polynomial identity

$$(x^{q^m} - x)^n = 0.$$

Hint. (i) The algebra R has a finite number of elements (why?). For a fixed $r \in R$, at least two of the elements $r, r^2, r^3, \ldots$ coincide. Hence there exist two different positive integers k and l such that the polynomial $f_r(x) = x^k - x^l$ satisfies $f_r(r) = 0$ in R. Since R is a finite set, the polynomial

$$g(x) = \prod_{r \in R} f_r(x) \in K\langle X\rangle$$

is well defined and is a polynomial identity for R.

(ii) Since $M_n(K)$ is a finite set, there exists a finite extension $\mathbb{F}_{q^m}$ of the base field $K = \mathbb{F}_q$ which contains the eigenvalues of all matrices r in $M_n(K)$. Consider r written in its Jordan normal form (over the field $\mathbb{F}_{q^m}$). Then $r^{q^m} - r$ is an upper triangular matrix and its n-th power is equal to 0.

Remark 2.1.11 Some important properties of associative algebras are expressed in the language of polynomial identities. We have seen this for the commutativity. Other examples come from nonunitary algebras. The algebra R is *nil of bounded index* if there exists an $n \in \mathbb{N}$ such that $x^n = 0$ is an identity for R; the algebra R is *nilpotent of class* $\leq n$ if $x_1 \ldots x_n = 0$ for R.

Remark 2.1.12 It turns out that the class of all PI-algebras has good structure and combinatorial properties similar to those of the commutative and the finite dimensional algebras (see e.g. the books by Rowen [231] and Kemer [142]). We hope to illustrate this claim in the present book.

Remark 2.1.13 Starting with the free Lie algebra $L(X)$ we can define the notion of polynomial identity for a Lie algebra G. The basic properties of Lie algebras with polynomial identities are similar to those of associative PI-algebras (see e.g. the book by Bahturin [21]). In the sequel we shall give the basic definitions for associative algebras. Nevertheless, comparing the theory of Lie algebras with polynomial identities with that of associative PI-algebras, there is a big difference. The deep structure and combinatorial properties of the associative algebras do not hold for Lie algebras. On the other hand, many problems are stated in the same way in both the associative and Lie cases and very often the same methods for investigation work.

Exercise 2.1.14 (i) Let G be the two-dimensional Lie algebra with basis as a vector space $\{a, b\}$ and multiplication $[a, b] = a$. Show that G satisfies the polynomial identity

$$[[x_1,x_2],[x_3,x_4]]=0 \ \ \text{(the } \textit{metabelian identity}\text{)}.$$

(ii) If G is a finite dimensional Lie algebra and $\dim G<n$, then G satisfies the *Lie standard identity of degree* $n+1$ (but in n skew-symmetric variables)

$$x_0s_n(\mathrm{ad}x_1,\ldots,\mathrm{ad}x_n)=\sum_{\sigma\in S_n}(\mathrm{sign}\sigma)[x_0,x_{\sigma(1)},\ldots,x_{\sigma(n)}]=0.$$

(iii) Show that the Lie algebra $(U_n(K))^{(-)}$ of all upper triangular $n\times n$ matrices satisfies the identity

$$[[x_1,x_2],\ldots,[x_{2n-1},x_{2n}]]=0.$$

Exercise 2.1.15 Let W_n be the set of all derivations of the polynomial algebra in n variables. (The linear operator δ of the vector space $K[x_1,\ldots,x_n]$ is a *derivation* if $\delta(uv)=\delta(u)v+u\delta(v)$, $u,v\in K[x_1,\ldots,x_n]$.)

(i) Show that W_n is a Lie algebra with respect to the operation $[\delta_1,\delta_2]=\delta_1\delta_2-\delta_2\delta_1$.

(ii) Show that W_1 satisfies the Lie standard identity

$$x_0s_4(\mathrm{ad}x_1,\mathrm{ad}x_2,\mathrm{ad}x_3,\mathrm{ad}x_4)=0.$$

(iii) Show that W_n satisfies some Lie standard identity.

Hint. (ii)–(iii) Every mapping $\{x_1,\ldots,x_n\}\longrightarrow K[x_1,\ldots,x_n]$ is uniquely extended to a derivation and this gives that the elements of W_n are of the form

$$\sum_{i=1}^{n}f_i(x_1,\ldots,x_n)\frac{\partial}{\partial x_i},\ f_i\in K[x_1,\ldots,x_n],$$

where $\partial/\partial x_i$ are the usual partial derivatives. For $n=1$ check that

$$\sum_{\sigma\in S_4}(\mathrm{sign}\sigma)\left[f_0(x)\frac{\partial}{\partial x},f_{\sigma(1)}(x)\frac{\partial}{\partial x},f_{\sigma(2)}(x)\frac{\partial}{\partial x},f_{\sigma(3)}(x)\frac{\partial}{\partial x},f_{\sigma(4)}(x)\frac{\partial}{\partial x}\right]=0$$

for $f_0,\ldots,f_4\in K[x]$. For further hints see the book by Bahturin [21].

Exercise 2.1.16 Let the base field K be finite and let the Lie algebra G be finite dimensional. Prove that G satisfies a polynomial identity of the form $xf(\mathrm{ad}y)=0$.

Remark 2.1.17 As in the case of associative algebras, some classical properties and results for Lie algebras can be stated in the language of polynomial identities. Let G be a Lie algebra.

(i) The algebra G is *abelian* if it satisfies the identity $[x_1,x_2]=0$ (i.e. has a trivial multiplication).

(ii) The algebra G is *nilpotent of class* $\leq n$ if it satisfies $[x_1, \ldots, x_{n+1}] = 0$. (Pay attention to the difference in the class of nilpotency of associative and Lie algebras and compare it with the class of nilpotency for groups. If one replaces the Lie commutators $[u, v] = uv - vu$ with group commutators $(u, v) = u^{-1}v^{-1}uv$, the definitions are the same for groups and Lie algebras.)

(iii) The algebra G is *solvable of class* $\leq n$ if it satisfies the identity $f_n = 0$, where $f_n = f_n(x_1, \ldots, x_{2^n})$, is defined inductively by

$$f_1(x_1, x_2) = [x_1, x_2],$$

$$f_n = [f_{n-1}(x_1, \ldots, x_{2^{n-1}}), f_{n-1}(x_{2^{n-1}+1}, \ldots, x_{2^n})],\ n > 1.$$

The solvable of class 2 Lie algebras are called *metabelian.*

(iv) A version of the Lie theorem (see e.g. [37]) gives that over a field of characteristic 0 any solvable finite dimensional Lie algebra satisfies the identity

$$[[x_1, x_2], \ldots, [x_{2n-1}, x_{2n}]] = 0$$

for some positive integer n.

2.2 Varieties and Relatively Free Algebras

Definition 2.2.1 Let $\{f_i(x_1, \ldots, x_{n_i}) \in K\langle X\rangle \mid i \in I\}$ be a set of polynomials in the free associative algebra $K\langle X\rangle$. The class $\mathfrak{V}$ of all associative algebras satisfying the polynomial identities $f_i = 0$, $i \in I$, is called the *variety* (*of associative algebras*) *defined* (or *determined*) *by the system of polynomial identities* $\{f_i \mid i \in I\}$. The variety $\mathfrak{W}$ is called a *subvariety of* $\mathfrak{V}$ if $\mathfrak{W} \subset \mathfrak{V}$. The set $T(\mathfrak{V})$ of all polynomial identities satisfied by the variety $\mathfrak{V}$ is called the *T-ideal* or the *verbal ideal* of $\mathfrak{V}$. We say, that the T-ideal $T(\mathfrak{V})$ *is generated as a T-ideal by the defining set of identities* $\{f_i \mid i \in I\}$ *of the variety* $\mathfrak{V}$. We use the notation $T(\mathfrak{V}) = \langle f_i \mid i \in I\rangle^T$ and say that the set $\{f_i \mid i \in I\}$ is a *basis of the polynomial identities for* $\mathfrak{V}$. The elements of $T(\mathfrak{V})$ are called *consequences* of the polynomial identities in the basis. If R is any algebra, we denote by $T(R)$ the T-ideal of the polynomial identities of R.

Exercise 2.2.2 Show that for any variety $\mathfrak{V}$ its T-ideal $T(\mathfrak{V})$ is a *fully invariant ideal* of $K\langle X\rangle$, i.e. $T(\mathfrak{V})$ is invariant under all endomorphisms of $K\langle X\rangle$. (In the notion “T-ideal”, “T” is for “Transformation” (= “Endomorphism”) and “Verbal” comes from group theory since the endomorphisms of the free group are defined by mapping the free generators to elements (words) of the free group.) Show that the same statement for fully invariance holds for the ideal $T(\mathfrak{V}) \cap K\langle x_1, \ldots, x_m\rangle$ of $K\langle x_1, \ldots, x_m\rangle$, where $x_1, \ldots, x_m \in X$.

Example 2.2.3 The class of all commutative algebras is a variety defined by the identity $[x_1, x_2] = 0$. The class of all associative algebras is also a variety defined by the empty set of polynomial identities.

Definition 2.2.4 For a fixed set Y, the algebra $F_Y(\mathfrak{V})$ in the variety $\mathfrak{V}$ is called a *relatively free algebra* of $\mathfrak{V}$ (or a $\mathfrak{V}$*-free algebra*), if $F_Y(\mathfrak{V})$ is free in the class $\mathfrak{V}$ (and is freely generated by Y).

Now we shall see that the relatively free algebras exist and two algebras of the same rank are isomorphic. In the sequel we shall denote the relatively free algebra of rank $m = |Y|$ by $F_m(\mathfrak{V})$. If Y is a countable infinite set we use the notation $F(\mathfrak{V})$ instead of $F_\infty(\mathfrak{V})$.

Proposition 2.2.5 *Let $\mathfrak{V}$ be a variety defined by $\{f_i \mid i \in I\}$, let Y be any set and let J be the ideal of $K\langle Y\rangle$ generated by*

$$\{f_i(g_1, \ldots, g_{n_i}) \mid g_j \in K\langle Y\rangle, \, i \in I\}.$$

Then the algebra $F = K\langle Y\rangle/J$ is a relatively free algebra in $\mathfrak{V}$ with set of free generators $\overline{Y} = \{y + J \mid y \in Y\}$. Every two relatively free algebras of the same rank in $\mathfrak{V}$ are isomorphic.

Proof. (i) First we shall see that $F \in \mathfrak{V}$. Let $f_i(x_1, \ldots, x_n)$ be one of the defining identities of $\mathfrak{V}$ and let $\bar{g}_1, \ldots, \bar{g}_n$ be arbitrary elements of F, $\bar{g}_j = g_j + J$, $g_j \in K\langle Y\rangle$. Then $f_i(g_1, \ldots, g_n) \in J$, hence $f_i(\bar{g}_1, \ldots, \bar{g}_n) = 0$ and this means that $f_i(x_1, \ldots, x_n) = 0$ is a polynomial identity for F. Hence $F \in \mathfrak{V}$.

(ii) Now we shall prove the universal property of F. Let R be any algebra in $\mathfrak{V}$ and let $\phi : \overline{Y} \longrightarrow R$ be an arbitrary mapping. We define a mapping $\theta : Y \longrightarrow R$ by $\theta(y) = \phi(\bar{y})$ and extend θ to a homomorphism (denoted also by θ) $\theta : K\langle Y\rangle \longrightarrow R$. This is always possible because $K\langle Y\rangle$ is the free associative algebra. In order to prove that ϕ can be extended to a homomorphism $F \longrightarrow R$, it is sufficient to show that $J \subseteq \mathrm{Ker}\theta$. Let $f \in J$, i.e.

$$f = \sum_{i \in I} u_i f_i(g_{i1}, \ldots, g_{in_i}) v_i, \; g_{ij}, u_i, v_i \in K\langle Y\rangle.$$

For arbitrary $r_1, \ldots, r_{n_i} \in R$, the element $f_i(r_1, \ldots, r_{n_i})$ is equal to zero in R and this implies that $\theta(f) = 0$, i.e. $J \subseteq \mathrm{Ker}\theta$ and $F \cong F_{\overline{Y}}(\mathfrak{V})$ is the relatively free algebra in $\mathfrak{V}$, freely generated by $\overline{Y}$.

(iii) Let $|Y| = |Z|$, $Y = \{y_i \mid i \in I\}$, $Z = \{z_i \mid i \in I\}$, and let $F_Y(\mathfrak{V})$ and $F_Z(\mathfrak{V})$ be the corresponding relatively free algebras. Since both algebras $F_Y(\mathfrak{V})$ and $F_Z(\mathfrak{V})$ are relatively free, we can define homomorphisms

$$\phi : F_Y(\mathfrak{V}) \longrightarrow F_Z(\mathfrak{V}), \; \psi : F_Z(\mathfrak{V}) \longrightarrow F_Y(\mathfrak{V})$$

by $\phi(y_i) = z_i$, $\psi(z_i) = y_i$. Since the compositions $\phi \circ \psi$ and $\psi \circ \phi$ act identically on Y and Z, respectively, we obtain that ϕ and ψ are isomorphisms.

Remark 2.2.6 It follows from the proof of Proposition 2.2.5 that the T-ideal of $K\langle X\rangle$ generated by $\{f_i \mid i \in I\}$ consists of all linear combinations of

$$u_i f_i(g_{i1}, \dots, g_{in_i}) v_i, \; g_{ij}, u_i, v_i, \in K\langle X\rangle .$$

Theorem 2.2.7 *There exists a* 1-1 *correspondence* π *between the T-ideals of* $K\langle X\rangle$ *and the varieties of associative algebras;* π *is a "Galois correspondence", i.e. for any two T-ideals* V_1, V_2 *the inclusion* $V_1 \subset V_2$ *is equivalent to* $\pi(V_1) \supset \pi(V_2)$.

Proof. For every T-ideal V we define $\mathfrak{V} = \pi(V)$ to be the variety determined by the polynomial identities belonging to V. This correspondence in "onto" because every variety has a T-ideal. Let $V_1 \neq V_2$ and $\pi(V_i) = \mathfrak{V}_i$, $i = 1, 2$. Then there exists a polynomial $f(x_1, \dots, x_n)$ which is in $V_1 \setminus V_2$ (or in $V_2 \setminus V_1$). Obviously, $f(x_1, \dots, x_n) = 0$ is a polynomial identity for $\mathfrak{V}_1$ and is not a polynomial identity for the relatively free algebra $F(\mathfrak{V}_2) = K\langle X\rangle / V_2 \in \mathfrak{V}_2$. Hence $\mathfrak{V}_1 \neq \mathfrak{V}_2$ and π is 1-1. Clearly, $\mathfrak{V}_1 \supset \mathfrak{V}_2$ if and only if every polynomial identity for $\mathfrak{V}_1$ is satisfied also by $\mathfrak{V}_2$, i.e. $T(\mathfrak{V}_1) \subset T(\mathfrak{V}_2)$.

Remark 2.2.8 If $\mathfrak{V}_1 \subset \mathfrak{V}_2$, then $T(\mathfrak{V}_1) \supset T(\mathfrak{V}_2)$ and we may consider the polynomial identities of $\mathfrak{V}_1$ modulo $T(\mathfrak{V}_2)$. Hence if we know the polynomial identities of $\mathfrak{V}_2$ and want to study the polynomial identities of $\mathfrak{V}_1$, we may work in the relatively free algebra $F(\mathfrak{V}_2)$.

2.3 The Theorem of Birkhoff

Definition 2.3.1 Let $\mathfrak{V}$ be a class of algebras. We denote by $\mathcal{C}\mathfrak{V}$, $\mathcal{S}\mathfrak{V}$ and $\mathcal{Q}\mathfrak{V}$, respectively, the classes obtained from $\mathfrak{V}$ by taking all Cartesian sums (i.e. "direct sums" allowing infinite support), subalgebras and factor algebras of algebras in $\mathfrak{V}$.

Theorem of Birkhoff 2.3.2 *A class of algebras* $\mathfrak{V}$ *is a variety if and only if* $\mathfrak{V}$ *is closed under taking Cartesian sums, subalgebras and factor algebras, i.e.* $\mathcal{C}\mathfrak{V}, \mathcal{S}\mathfrak{V}, \mathcal{Q}\mathfrak{V} \subseteq \mathfrak{V}$.

Proof. (i) Let $\mathfrak{V}$ be a variety and let $R_j \in \mathfrak{V}$, $j \in J$. The Cartesian sum $R = \sum_{j \in J} R_j$ consists of all "sequences" $(r_j \mid j \in J)$, $r_j \in R_j$, with coordinatewise defined operations. Let $f(x_1, \dots, x_n)$ be a polynomial identity for $\mathfrak{V}$. If $r^{(1)}, \dots, r^{(n)} \in R$, $r^{(i)} = (r_j^{(i)} \mid j \in J)$, then

$$f(r^{(1)}, \dots, r^{(n)}) = (f(r_j^{(1)}, \dots, r_j^{(n)}) \mid j \in J).$$

Clearly, all coordinates of $f(r^{(1)}, \dots, r^{(n)})$ are equal to 0 and $R \in \mathfrak{V}$. The proof of $\mathcal{S}\mathfrak{V}, \mathcal{Q}\mathfrak{V} \subseteq \mathfrak{V}$ is similar.

(ii) Let $\mathcal{C}\mathfrak{V}, \mathcal{S}\mathfrak{V}, \mathcal{Q}\mathfrak{V} \subseteq \mathfrak{V}$ and let $T(\mathfrak{V}) \subset K\langle X\rangle$ be the set of polynomial identities satisfied by the algebras in $\mathfrak{V}$. We denote by $\mathfrak{W}$ the variety defined by the identities from $T(\mathfrak{V})$. Obviously, $\mathfrak{V} \subseteq \mathfrak{W}$. We shall show the equality $\mathfrak{V} = \mathfrak{W}$. Let m be an arbitrary cardinality and let $Y = \{y_i \mid i \in I\}$ be a set with m elements. Let N be the set of all elements $f(x_1, \ldots, x_n)$ of $K\langle X\rangle$ which *are not* polynomial identities for $\mathfrak{V}$. We present $K\langle Y\rangle$ as a disjoint union of two subsets $T_m(\mathfrak{V})$ and N_m in the following way. Let $f(y_{i_1}, \ldots, y_{i_n})$ be any element of $K\langle Y\rangle$, where $y_{i_1}, \ldots, y_{i_n}$ are n different elements of Y (and $n \leq m$). If $f(x_1, \ldots, x_n) \in T(\mathfrak{V})$, then we assume that $f(y_{i_1}, \ldots, y_{i_n}) \in T_m(\mathfrak{V})$ and if $f(x_1, \ldots, x_n)$ is not a polynomial identity for $\mathfrak{V}$, i.e. $f(x_1, \ldots, x_n) \in N$, then $y_{i_1}, \ldots, y_{i_n} \in N_m$. For every $f = f(y_{i_1}, \ldots, y_{i_n}) \in N_m$ there exists an algebra $R_f \in \mathfrak{V}$ and elements $r_{i_1 f}, \ldots, r_{i_n f} \in R_f$, such that $f(r_{i_1 f}, \ldots, r_{i_n f}) \neq 0$ in R_f. For any $i \in I$, $i \neq i_1, \ldots, i_n$, we choose arbitrary elements $r_{if} \in R_f$ and define elements

$$z_i = (r_{if} \mid i \in I) \in \sum_{f \in N_m} R_f .$$

Since $\mathcal{C}\mathfrak{V}, \mathcal{S}\mathfrak{V} \subseteq \mathfrak{V}$, we obtain that the algebra F generated by z_i in $\sum_{f \in N_m} R_f$ belongs to $\mathfrak{V}$. On the other hand, if $g(y_{i_1}, \ldots, y_{i_n}) \in N_m$, then $g(z_{i_1}, \ldots, z_{i_n}) \neq 0$, because $g(r_{i_1 g}, \ldots, y_{i_n g}) \neq 0$ for any $g \in N_m$. Hence the kernel of the canonical homomorphism $K\langle Y\rangle \longrightarrow F$ extending $y_i \rightarrow z_i$, $i \in I$, coincides with $T_m(\mathfrak{V})$ and F is isomorphic to $F_m(\mathfrak{W})$, the relatively free algebra of rank m in $\mathfrak{W}$. Finally, since $\mathcal{Q}\mathfrak{V} \subseteq \mathfrak{V}$, and every m-generated algebra in $\mathfrak{W}$ is a homomorphism image of $F_m(\mathfrak{W})$ which is in $\mathfrak{V}$, we obtain that $\mathfrak{W} \subseteq \mathfrak{V}$, i.e. $\mathfrak{V} = \mathfrak{W}$.

Definition 2.3.3 For a class of algebras $\mathfrak{V}$ or for an algebra R, we denote by $\mathrm{var}\mathfrak{V}$ or by $\mathrm{var}R$ the variety of algebras defined by the polynomial identities from $T(\mathfrak{V})$ or $T(R)$ and call this variety the *variety generated by* $\mathfrak{V}$ *or* R. Instead of $F_m(\mathrm{var}R)$, sometimes we shall denote the relatively free algebras of $\mathrm{var}R$ by $F_m(R)$.

The following corollary is a consequence of the proof of Theorem of Birkhoff 2.3.2.

Corollary 2.3.4 *If* $\mathfrak{V}$ *is a class of associative algebras, then* $\mathrm{var}\mathfrak{V} = \mathcal{QSC}\mathfrak{V}$.

Exercise 2.3.5 Let R be a finite algebra (i.e. $|K| < \infty$ and $\dim R < \infty$). Show that $\mathrm{var}R$ is *locally finite*, i.e. every finitely generated algebra in $\mathrm{var}R$ is finite. Prove that $|F_m(\mathrm{var}R)| \leq |R|^{|R|^m}$, $m \in \mathbb{N}$.

Hint. Show that in the proof of Theorem of Birkhoff 2.3.2, in the construction of the relatively free algebra $F_m(\mathrm{var}R)$ it is sufficient to consider direct sums of $|R|^m$ copies of R instead of Cartesian sums.

Exercise 2.3.6 If $K = \mathbb{F}_q$ is the field with q elements, show that the polynomial identities

$$[x, y] = 0, \; x^{q^n} - x = 0$$

form a basis for the polynomial identities of the K-algebra $\mathbb{F}_{q^n}$.

Hint. Use that for any $0 \neq f(x_1, \ldots, x_m) \in K[X]/(x^{q^n} - x \mid x \in X)$, there exist some $\alpha_i \in \mathbb{F}_{q^n}$ such that $f(\alpha_1, \ldots, \alpha_m) \neq 0$. Hence the algebra $K[X]/(x^{q^n} - x \mid x \in X)$ is relatively free in the variety defined by the identities $[x, y] = 0$ and $x^{q^n} - x = 0$.

Exercise 2.3.7 Show that the polynomial identity $x^2 - x = 0$ implies the commutativity.

Hint. Use that

$$0 = (x + y)^2 - (x + y) = (x^2 - x) + (y^2 - y) + (xy + yx) = xy + yx$$

and obtain the anticommutative law. For $x = y$ it gives $2x^2 = 0$ and, since $x^2 = x$, we see that $2x = 0$. Therefore $x = -x$ and the anticommutativity is equivalent to the commutativity.

Remark 2.3.8 It is known that the polynomial identity $x^n - x = 0$ implies the commutativity, i.e. in Exercise 2.3.6 one needs only $x^{q^n} - x = 0$. See the book by Herstein [126] for the proof.

Remark 2.3.9 In a similar way, as in Definition 2.2.1, starting, respectively, with the free Lie algebra and the free group, one can introduce polynomial identities and varieties of Lie algebras, identities and varieties of groups, etc. and to prove Theorem of Birkhoff 2.3.2 and Corollary 2.3.4. See the books by Bahturin [21] and Hanna Neumann [192] for details.

3. The Specht Problem

One of the reasons for introducing the notion of varieties of groups and algebras is that the varieties give some rough classification of all groups and algebras in the language of identities. Of course this classification is very rough. For example, as we shall see in Exercise 4.3.7, the only variety of commutative algebras over an infinite field is the variety of all commutative algebras. From this point of view, commutative algebra is "trivial". Since we want to classify all varieties, the first question is whether we can do this in finite terms. In this chapter we consider the Specht problem, whether every variety of algebras can be defined by a finite system of polynomial identities. Together with a similar problem in group theory, this has been one of the main problems in the theory of varieties of groups and algebras for more than 30 years. Even now, it is still open for some classes of groups and algebras. Here we give an example of a variety of Lie algebras over a field of characteristic 2 which has no finite basis of its polynomial identities.

3.1 The Finite Basis Property

Definition 3.1.1 A variety of associative algebras $\mathfrak{V}$ is called *finitely based* (or has a *finite basis of its polynomial identities*) if $\mathfrak{V}$ can be defined by a finite system of polynomial identities (from the free associative algebra $K\langle X\rangle$, $X = \{x_1, x_2, \ldots\}$). If $\mathfrak{V}$ cannot be defined by a finite system of identities, it is called *infinitely based.* If all subvarieties of $\mathfrak{V}$, including $\mathfrak{V}$ itself, are finite based, $\mathfrak{V}$ satisfies the *Specht property.* One defines similarly the finite/infinite basis and Specht properties for varieties of groups, Lie algebras, etc.

The following problem was asked first for groups by B.H. Neumann in his thesis in 1935 (see also his paper [191] in 1937 and the book by Hanna Neumann [192] for references), then by Specht [245] in 1950 for associative algebras over a field of characteristic 0. Now it is known as the *Specht problem.*

Specht Problem 3.1.2 Is every variety of associative or Lie algebras finitely based?

The investigations on the Specht problem are in two directions:

(i) *To show that some varieties satisfy the Specht property.*

(ii) *To construct counterexamples to the Specht problem, i.e. examples of varieties which have ho finite basis for their identities.*

The first significant result in direction (i) is due to Sheila Oates and Powell [197] who established that a variety generated by a finite group has the Specht property. Later, the method was extended to the case when the variety is generated by a finite ring with some reasonable conditions on the ring (e.g. for Lie rings, associative rings, etc.), see the book by Bahturin [21] for details.

Theorem 3.1.3 *Let R be a finite group, a finite associative or Lie ring, a finite dimensional associative or Lie algebra over a finite field. Then* $\mathrm{var}R$ *has the Specht property.*

Another group of results shows that the variety has the Specht property if it satisfies an identity of some special kind. Finally, in a series of papers, Kemer (see his book [142] for details) developed a powerful and complicated technique which allowed him to show that the T-ideals of the free associative algebras over a field of characteristic 0 have many properties of the ideals of the polynomial algebras with commuting variables. In particular, in 1987 Kemer gave a positive solution of the Specht problem for associative algebras over a field of characteristic 0.

Theorem 3.1.4 (Kemer) *Every variety of associative algebras over a field of characteristic* 0 *has a finite basis for its polynomial identities.*

Concerning (ii), the first counterexamples were given in group theory by Olshanskii [201] in 1970 and then by Adyan [5] and Vaughan-Lee [258], also in 1970. Maybe the simplest example is due to Kleiman [149] and Bryant [39] who showed that *the system of group identities*

$$(x_1^2\dots x_n^2)^4 = 1,\ n = 1, 2, \dots,$$

does not follow from any of its finite subsystems. This system of identities defines the variety which is usually denoted by $\mathfrak{B}_4\mathfrak{B}_2$ and consists of all groups G such that the normal subgroup H of G generated by all squares is of exponent 4.

For Lie algebras the first counterexample was given by Vaughan-Lee [259] in 1970 for Lie algebras over a field of characteristic 2 and then in 1974 by Drensky [69] and Kleiman (unpublished) for Lie algebras over a field of any positive characteristic. It was also shown in the papers by Vaughan-Lee and Drensky that Theorem 3.1.3 cannot be generalized to the case of finite dimensional Lie algebras over an infinite field of positive characteristic. In particular, the result of Vaughan-Lee [259] (as precised in the Ph.D. thesis of Drensky [70] for (ii)) is the following.

Theorem 3.1.5 *Let K be a field of characteristic 2.*

(i) *Let $\mathfrak{V}$ be the variety of Lie algebras defined by the polynomial identities*

$$[[[x_1,x_2],[x_3,x_4]],x_5]=0 \ (\textit{the centre-by-metabelian identity}),$$

$$[[x_1,x_2,x_3,\ldots,x_n],[x_1,x_2]]=0,\ n=3,4,\ldots$$

Then $\mathfrak{V}$ has no finite basis of its polynomial identities.

(ii) *Let K be infinite and let $\mathfrak{M}_2$ be the variety of Lie algebras generated by the Lie algebra $M_2(K)^{(-)}$ of 2×2 matrices with entries from K. Then $\mathfrak{M}_2$ has no finite basis for its identities and the polynomial identities in (i) together with the polynomial identities*

$$[[x_1,x_4,x_5,\ldots,x_n],[x_2,x_3]]+[[x_2,x_4,x_5,\ldots,x_n],[x_3,x_1]]+$$

$$[[x_3,x_4,x_5,\ldots,x_n],[x_1,x_2]]=0,\ n=4,5,\ldots,$$

form an infinite basis of the identities for $\mathfrak{M}_2$.

Another example of a variety of Lie algebras over a field of characteristic 2 without finite basis of its polynomial identities was given also by Vaughan-Lee [260] in 1975. His proof is based on work in the universal enveloping algebra of the free nilpotent of class 2 Lie algebra. We forward to the book by Bahturin [21] for a further survey on the Specht problem for Lie algebras.

Concerning finite nonassociative algebras, Polin [207] constructed a simple example showing that Theorem 3.1.3 does not hold for arbitrary finite rings and algebras.

Up till now, the Specht problem is still open for associative algebras over a field of positive characteristic and for Lie algebras over a field of characteristic 0.

3.2 Lie Algebras in Characteristic 2

The main purpose of this section is to give a proof of Theorem 3.1.5 (i). We start with some constructions of Lie algebras.

Exercise 3.2.1 Let G be a Lie algebra and let $\mathcal{D}(G)$ be the vector space of all derivations of G, i.e. linear operators of the vector space G which satisfy

$$\delta([u,v])=[\delta(u),v]+[u,\delta(v)],\ \delta\in\mathcal{D}(G),u,v\in G.$$

(i) Show that $\mathcal{D}(G)$ is a subalgebra of the algebra $(\mathrm{End}_K G)^{(-)}$, the Lie algebra of all linear operators of G. (Compare with Exercise 2.1.15 (i).)

(ii) Let $D\subset\mathcal{D}(G)$ be a Lie algebra of derivations of G and let the algebra H be defined by $H=G\oplus D$ as a vector space, with multiplication given by

$$[g_1+\delta_1,g_2+\delta_2]=([g_1,g_2]+\delta_1(g_2)-\delta_2(g_1))+[\delta_1,\delta_2],\ g_i\in G,\delta_i\in D,\ i=1,2.$$

Show that H is a Lie algebra and G is an ideal of H.

Exercise 3.2.2 Let G be a Lie algebra and $u \in G$. Show that

$$\mathrm{ad}u : v \to [v, u],\ v \in G,$$

is a derivation of G. Such a derivation is called *inner.*

Till the end of the proof of our main result Theorem 3.1.5 (i) we assume that $\mathrm{char}K = 2$. Therefore, for any Lie algebra G, we obtain that $g = -g$ and $[g_2, g_1] = [g_1, g_2]$, $g, g_1, g_2 \in G$.

Lemma 3.2.3 *Let $n \in \mathbb{N}$ be fixed and let*

$$C_n = K[t_0, t_1, \ldots, t_n]/(t_0^3, t_1^2, \ldots, t_n^2).$$

Let G_n be the algebra with basis as a vector space

$$\{u(t_0, t_1, \ldots, t_n)a,\ t_0^2 t_1 \ldots t_n b \mid u \text{ a monomial with coefficient 1 in } C_n\}.$$

Assuming that $u_1 u_2 b = 0$ if $u_1 u_2 \neq t_0^2 t_1 \ldots t_n$, we define the multiplication between the basis elements of G_n by

$$[u_1 a, u_2 a] = u_1 u_2 b,$$

and all other products of the basis elements are 0. Then G_n is a Lie algebra which is nilpotent of class 2 (i.e. satisfies the identity $[x_1, x_2, x_3] = 0$).

Proof. Since the product of any three elements of the algebra G_n is equal to 0, we obtain that G_n satisfies the identity $[x_1, x_2, x_3] = 0$ and the Jacobi identity. We shall complete the proof if we establish the anticommutativity, which is equivalent to show that

$$[u_1 a, u_2 a] = -[u_2 a, u_1 a],\ [ua, ua] = 0$$

for any monomials $u_1, u_2, u \in C_n$. Since $\mathrm{char}K = 2$, the definition of the multiplication between $u_1 a$ and $u_2 a$ gives the first equality. The second equality $[ua, ua] = 0$ is also obvious because $n > 0$ and $u^2 \neq t_0^2 t_1 \ldots t_n$.

Lemma 3.2.4 *In the notation of Lemma 3.2.3, let $d_0, d_1, \ldots, d_n$ be the linear operators of G_n defined on the basis vectors of G_n by*

$$d_i(ua) = t_i ua,\ d_i(ub) = 0,\ i = 0, 1, \ldots, n.$$

Then the vector space $D_n = \mathrm{span}\{d_0, d_1, \ldots, d_n\}$ spanned by $d_0, d_1, \ldots, d_n$ is an abelian (i.e. commutative) Lie algebra of derivations of G_n.

Proof. Clearly, d_i, d_j, $i, j = 0, 1, \ldots, n$, are commuting linear operators of G_n. It is sufficient to see that each d_i acts as a derivation on the products of basis elements of G_n. The only nontrivial case is $d_i([u_1a, u_2a])$ (why?). Direct calculations show

$$d_i([u_1a, u_2a]) = d_i(u_1u_2b) = 0,$$

$$[d_i(u_1a), u_2a] + [u_1a, d_i(u_2a)] = ((t_iu_1)u_2)b + (u_1(t_iu_2))b = 0,$$

$$d_i([u_1a, u_2a]) = [d_i(u_1a), u_2a] + [u_1a, d_i(u_2a)] = 0,$$

and this means that d_i is a derivation. Since $[d_i, d_j] = 0$, the vector space D_n is an abelian Lie algebra.

Lemma 3.2.5 *In the notation of the previous two lemmas, let $H_n = G_n \oplus D_n$ be the Lie algebra obtained as in Exercise 3.2.1 (ii). Then:*

(i) *The algebra H_n satisfies the central-by-metabelian polynomial identity*

$$[[x_1, x_2], [x_3, x_4], x_5] = 0.$$

(ii) *The centre of H_n (i.e. the set $C(H_n)$ of all elements $z \in H_n$ such that $[H_n, z] = 0$) coincides with the vector space spanned on $t_0^2t_1 \dots t_nb$.*

Proof. (i) One can directly see that the *commutator ideal* $H_n' = [H_n, H_n]$ is contained in G_n and $G_n' = Kt_0^2t_1 \dots t_nb$. This implies that

$$H_n'' = (H_n')' \subset G_n' = Kt_0^2t_1 \dots t_nb.$$

The definition of the multiplication in H_n shows that the element $t_0^2t_1 \dots t_nb$ is in the centre of H_n. This gives that $[[H_n, H_n], [H_n, H_n], H_n] = 0$, i.e. H_n is centre-by-metabelian. We leave part (ii) of the lemma for exercise.

Proposition 3.2.6 *The algebra H_n satisfies the polynomial identities*

$$f_k(x_1, \dots, x_k) = [[x_1, x_2, x_3, \dots, x_k], [x_1, x_2]] = 0,\ k \geq 3, k \neq n+2,$$

and $f_{n+2}(x_1, \dots, x_{n+2})$ does not vanish on H_n.

Proof. In order to show that $f_{n+2} \neq 0$ in H_n, we replace the variables by $\bar{x}_1 = a$, $\bar{x}_2 = d_0$, $\bar{x}_i = d_{i-2}$ for $i = 3, \dots, n+2$. Direct calculations show that

$$f_{n+2}(\bar{x}_1, \dots, \bar{x}_{n+2}) = t_0^2t_1 \dots t_nb \neq 0,$$

and $f_{n+2} \neq 0$ in H_n. Now, let $k \neq n+2$ and let us assume that $f_k(\bar{x}_1, \dots, \bar{x}_k) \neq 0$ for some $\bar{x}_i \in H_n$. Since $f_k(\bar{x}_1, \dots, \bar{x}_k)$ is linear in each of the variables $\bar{x}_i$, $i = 3, \dots, k$, without loss of generality we may assume that $\bar{x}_i$, $i = 3, \dots, k$, are some of the basis elements of H_n. Additionally, we may assume that

$$[\bar{x}_1, \bar{x}_2] = c_1a + c_2b,\ a, b \in C_n,$$

is some element of G_n. It is easy to see that the only elements which give a nonzero contribution to $f_k(\bar{x}_1, \dots, \bar{x}_k)$ are $[\bar{x}_1, \bar{x}_2] = ca$, where $c \in C_n$ is a polynomial without constant term, $\bar{x}_i = d_{j_i}$, $i = 3, \ldots, k$. Hence, we may assume that

$$f_k(\bar{x}_1, \ldots, \bar{x}_k) = [ca, d_{i_3}, \ldots, d_{i_k}, ca] = c^2 t_{i_3} \ldots t_{i_k} b \neq 0.$$

Let $c = \sum \alpha_j u_j$, where u_j are monomials of C_n and $\alpha_j \in K$. The associative algebra C_n is commutative and $\mathrm{char} K = 2$. Hence

$$c^2 = (\sum \alpha_j u_j)^2 = \sum \alpha_j^2 u_j^2,$$

and this means that it is sufficient to assume that $[\bar{x}_1, \bar{x}_2] = ua$ for some monomial u of degree ≥ 1 in C_n. Now

$$f_k(\bar{x}_1, \ldots, \bar{x}_k) = u^2 t_{i_3} \ldots t_{i_k} b \neq 0.$$

The nonzero elements $fb \in H_n$ are multiple of $t_0^2 t_1 \ldots t_n b$ and we obtain that

$$u^2 t_{i_3} \ldots t_{i_k} = t_0^2 t_1 \ldots t_n.$$

In this way we obtain that $u = t_0$ (since $\mathrm{deg} u > 0$) and $k = n + 2$, which is impossible. Therefore, $f_k(\bar{x}_1, \ldots, \bar{x}_k) = 0$ in H_n.

Theorem 3.2.7 (Vaughan-Lee [259]) *Let K be a field of characteristic 2 and let $\mathfrak{V}$ be the variety of Lie algebras defined by the polynomial identities*

$$c(x_1, x_2, x_3, x_4, x_5) = [[[x_1, x_2], [x_3, x_4]], x_5] = 0,$$

$$f_n = [[x_1, x_2, x_3, \ldots, x_n], [x_1, x_2]] = 0,\ n = 3, 4, \ldots$$

Then $\mathfrak{V}$ has no finite basis of its polynomial identities.

Proof. Let $\mathfrak{V}$ be defined by a finite system of polynomial identities

$$g_i(x_1, \ldots, x_m) = 0,\ i = 1, 2, \ldots, k.$$

All g_i belong to the T-ideal $V = T(\mathfrak{V})$ generated in the free Lie algebra $L(X)$ by

$$c(x_1, x_2, x_3, x_4, x_5),\ f_n(x_1, \ldots, x_n),\ n = 3, 4, \ldots,$$

and V is generated by g_i, $i = 1, \ldots, k$. Clearly, each g_i depends on a finite number of f_n's and, since the number of the g_i's is finite, we obtain that as a T-ideal V is generated by $c(x_1, x_2, x_3, x_4, x_5)$ and a finite number of f_n's, say $f_3, \ldots, f_k$. Hence, every centre-by-metabelian Lie algebra H which satisfies the identities $f_n = 0$, $n \leq k$, satisfies also all other identities $f_n = 0$, $n > k$. This contradicts with Proposition 3.2.6 and completes the proof.

The existence of an infinitely based variety allows to construct different examples.

Corollary 3.2.8 *Over a field of characteristic* 2*, there exist continually many varieties of Lie algebras.*

Proof. Let S be any subset of $\mathbb{N} \setminus \{1, 2\}$. We define a variety $\mathfrak{V}_S$ of centre-by-metabelian Lie algebras (i.e. we assume that $\mathfrak{V}_S$ satisfies

$$[[x_1, x_2], [x_3, x_4], x_5] = 0)$$

by the polynomial identities

$$f_k(x_1, \ldots, x_k) = [[x_1, x_2, x_3, \ldots, x_k], [x_1, x_2]] = 0,\ k \in S.$$

Let $S \neq S'$ and let, for example, $k \in S \setminus S'$. Proposition 3.2.6 gives that the algebra H_{k-2} does not satisfy the identity $f_k = 0$. Hence H_{k-2} does not belong to $\mathfrak{V}_S$. From the other hand, H_{k-2} satisfies all other polynomial identities $f_n = 0$, $n = 3, 4, \ldots$, hence $H_k \in \mathfrak{V}_{S'}$. In this way all varieties $\mathfrak{V}_S$ are different. Since the subsets of $\mathbb{N} \setminus \{1, 2\}$ are continually many, we obtain the statement of the corollary.

Corollary 3.2.9 *Over a field of characteristic* 2*, there exist continually many varieties of Lie algebras which form a chain with respect to the inclusion.*

Proof. We use a trick of Olshanskii [202]. Let $\mathbb{Q} = \{r_1, r_2, \ldots\}$ be the set of the rationals. For any $\alpha \in \mathbb{R}$ we construct a variety $\mathfrak{W}_\alpha$ consisting of all centre-by-metabelian algebras satisfying the polynomial identities

$$f_k(x_1, \ldots, x_k) = [[x_1, x_2, x_3, \ldots, x_k], [x_1, x_2]] = 0,\ r_k \geq \alpha.$$

As in Corollary 3.2.8, we see that if $\alpha < \beta$, then $\mathfrak{W}_\alpha$ is a proper subvariety of $\mathfrak{W}_\beta$ (because $\mathfrak{W}_\alpha$ satisfies more identities than $\mathfrak{W}_\beta$).

Exercise 3.2.10 Show that over any field the *free centre-by-metabelian Lie algebra* (sometimes one uses the notation $F([\mathfrak{A}^2, \mathfrak{E}]) = L/[L'', L]$, where $L = L(X)$) is spanned as a vector space by

$$[x_{i_1}, x_{i_2}, \ldots, x_{i_m}],\ [[x_{j_1}, x_{j_2}, \ldots, x_{j_n}], [x_{k_1}, x_{k_2}]],$$

where $i_1 > i_2 \leq i_3 \leq \ldots i_m$, $j_1 > j_2 \leq j_3 \leq \ldots \leq j_n$, $k_1 > k_2$, $m = 1, 2, \ldots$, $n = 2, 3, \ldots$

Hint. See the paper by Vaughan-Lee [259] or the book by Bahturin [21]. Another possibility is to try to solve the exercise after reading Chapter 5.

Exercise 3.2.11 In the case of characteristic 2, show that there exists an $m \in \mathbb{N}$ such that the algebras H_n defined in Lemma 3.2.5 satisfy the *Engel identity*

$$x_2(\mathrm{ad}^m x_1) = [x_2, \underbrace{x_1, \ldots, x_1}_{m \text{ times}}] = 0.$$

Hint. Use that in H_n the only nonzero contribution in

$$[[h_1, h_2](\mathrm{ad}^s h), h_3],\ h, h_1, h_2, h_3 \in H_n,$$

comes from $h \in D_n$ and use that

$$\mathrm{ad}^2(h' + h'') = \mathrm{ad}^2 h' + \mathrm{ad}^2 h'',\ h', h'' \in D_n.$$

Exercise 3.2.12 Derive from Exercises 3.2.10 and 3.2.11 that the varieties in Corollaries 3.2.8 and 3.2.9 can be chosen *locally nilpotent* (i.e. their finitely generated algebras are nilpotent) and hence *locally finite dimensional.*

Hint. See the book by Bahturin [21] or the paper of Drensky [73] for further comments.

Exercise 3.2.13 Let $\mathrm{char} K = 2$ and let $U = U(F(\mathfrak{N}_2))$ be the universal enveloping algebra of the free nilpotent of class 2 Lie algebra $F(\mathfrak{N}_2)$. Let J be the minimal ideal of U containing the elements

$$f_n = [x_1, x_2][x_2, x_3] \dots [x_{n-1}, x_n][x_n, x_1] \in U,\ n = 2, 3, \dots,$$

and such that J is closed under all linear transformations, i.e. $u(x_1, \dots, x_n) \in J$ implies that $u(\sum \alpha_{i1} x_i, \dots, \sum \alpha_{in} x_i) \in J$ for any $\alpha_{ij} \in K$. Show that J is not finitely generated as an ideal invariant under linear transformations. Derive from here, that the variety of Lie algebras over a field of characteristic 2 defined by the polynomial identities

$$[[x_1, x_2, x_3], [x_4, x_5, x_6]] = 0,$$

(this identity defines the variety $\mathfrak{A}\mathfrak{N}_2$ of *abelian-by-nilpotent of class two* Lie algebras) and

$$g_n = [[y_1, y_2, y_3], [x_1, x_2], [x_2, x_3], \dots, [x_{n-1}, x_n], [x_n, x_1]] = 0,\ n = 2, 3, \dots,$$

is not finitely based.

Hint. This is the result of Vaughan-Lee [260] mentioned above. See the original paper for help.

Remark 3.2.14 If $\mathrm{char} K = 0$, the previous exercise is not true. In this case Volichenko [264] showed that the ideals of $U = U(F(\mathfrak{N}_2))$ invariant under linear transformations are finitely generated in the class of all such ideals. It was one of the main steps of his proof that every subvariety of the variety $\mathfrak{A}\mathfrak{N}_2$ is finitely based.

Problem 3.2.15 Let $\mathrm{char} K = p > 0$. Are the subvarieties of $\mathfrak{A}\mathfrak{N}_2$ finitely based? Are the subvarieties of $\mathfrak{A}\mathfrak{N}_p$ finitely based? (The variety $\mathfrak{A}\mathfrak{N}_p$ is defined by

$$[[x_1, x_2, \dots, x_{p+1}], [y_1, y_2, \dots, y_{p+1}]] = 0.)$$

Very probably the answer to the second question is negative.

Problem 3.2.16 Let K be an infinite field of odd characteristic p. Does the Lie algebra $M_p(K)^{(-)}$ have a finite basis for its polynomial identities? Very probably the answer is negative.

Problem 3.2.17 Let K be an infinite field of characteristic 2. Does the associative algebra $M_2(K)$ have a finite basis for its polynomial identities? This algebra is considered as the most promising candidate for the negative solution of the Specht problem for associative algebras over a field of positive characteristic. No guess whether the answer is positive or negative.

Exercise 3.2.18 Let V_2 be the 2-dimensional vector space with a fixed basis e_1, e_2 and let $U_2(K)$ be the subalgebra of upper triangular matrices in $M_2(K)$. Assuming that $U_2(K)$ and $M_2(K)$ act on V_2 from the right, define nonassociative algebras

$$R_1 = V_2 \oplus U_2(K), \ R_2 = V_2 \oplus M_2(K)$$

as vector spaces and with multiplication

$$(v_1 + a_1)(v_2 + a_2) = v_1 a_2, \ v_i \in V_2, a_i \in M_2(K), \ i = 1, 2.$$

Show that R_1 and R_2 are *left nilpotent*, i.e. $x_1(x_2x_3) = 0$ and have no finite basis for their polynomial identities.

Hint. The algebra R_2 is the example of Polin [207] and the proof for the infinite basis property for R_1 is similar, based on the fact that the ideal of the polynomial identities of $U_2(K)$ is not finitely generated as a left ideal invariant under linear transformations. See the paper by Polin [207] for more help.

4. Numerical Invariants of T-Ideals

This chapter is devoted to the reduction of arbitrary polynomial identities to polynomial identities of special form: homogeneous, multilinear and the so called proper polynomial identities. We also introduce the most important numerical invariants of T-ideals: the codimension sequence of the T-ideal and the Hilbert series of the corresponding relatively free algebras. We find various relations between the ordinary and the proper numerical invariants. The considered reductions turn to be very useful both for studying the general properties of PI-algebras and in the investigation of the polynomial identities of concrete important algebras.

4.1 Graded Vector Spaces

Definition 4.1.1 The vector space V is *graded* if it is a direct sum of its subspaces $V^{(n)}$, $n \geq 0$, i.e.

$$V = \bigoplus_{n\geq 0} V^{(n)} = \sum_{n\geq 0}{}^{\oplus} V^{(n)}.$$

The subspaces $V^{(n)}$ are called the *homogeneous components of degree* n *of* V. If no misunderstanding, we shall write $\sum$ instead of $\sum^{\oplus}$. Similarly, we introduce a *multigrading* on V if

$$V = \sum_{i=1}^{m} \sum_{n_i\geq 0} V^{(n_1,\ldots,n_m)}$$

and call $V^{(n_1,\ldots,n_m)}$ its *homogeneous component of degree* $(n_1,\ldots,n_m)$. The subspace W of the graded vector space $V = \sum_{n\geq 0} V^{(n)}$ is a *graded* or (*homogeneous*) *subspace* if $W = \sum_{n\geq 0}(W \cap V^{(n)})$. In this case, the factor space V/W can also be naturally graded (and we say that V/W *inherits the grading of* V).

Example 4.1.2 (i) The polynomial algebra $K[x_1,\ldots,x_m]$ is graded assuming that the homogeneous polynomials of degree n (in the usual sense) are the homogeneous elements of degree n. Similarly, $K[x_1, \ldots, x_m]$ has a multigrading, counting the entry of each variable in the monomials. Analogously, one can define grading and multigrading on the free associative algebra $K\langle x_1, \ldots, x_m\rangle$ and a grading on $K\langle X\rangle$. Usually we shall assume that $K[x_1, \ldots, x_m]$ and $K\langle x_1, \ldots, x_m\rangle$ are equipped with these two gradings.

(ii) The Grassmann algebra has a grading inherited from the grading of $K\langle X\rangle$.

Definition 4.1.3 Let $V = \sum_{n\geq 0} V^{(n)}$ be a graded vector space and let $\dim V^{(n)} < \infty$. The formal power series

$$H(V,t) = \mathrm{Hilb}(V,t) = \sum_{n\geq 0} \dim V^{(n)} t^n$$

is called the *Hilbert* (or *Poincaré*) *series* of V. For a function $f(t)$, we make the usual convention that $H(V,t) = f(t)$ if the series $H(V,t)$ converges in some neighbourhood of 0 and the functions $H(V,t)$ and $f(t)$ are equal there. Similarly, if the vector space

$$V = \sum_{n} V^{(n_1,\ldots,n_m)},\ n = (n_1, \ldots, n_m),$$

is multigraded, then the *Hilbert series* of V is

$$H(V,t_1,\ldots,t_m) = \mathrm{Hilb}(V,t_1,\ldots,t_m) = \sum_{n} \dim V^{(n_1,\ldots,n_m)} t_1^{n_1}\ldots t_m^{n_m}.$$

Exercise 4.1.4 If $V = \sum V^{(n)}$ and $W = \sum W^{(n)}$ are graded vector spaces with the same (multi)grading, show that $V \otimes W$ is also graded assuming that its homogeneous components are

$$(V\otimes W)^{(n)} = \sum_{n'+n''=n} V^{(n')} \otimes W^{(n'')}.$$

Exercise 4.1.5 Considering graded vector spaces, prove that

$$\mathrm{Hilb}(V\oplus W,t) = \mathrm{Hilb}(V,t) + \mathrm{Hilb}(W,t),$$

$$\mathrm{Hilb}(V\otimes W,t) = \mathrm{Hilb}(V,t)\mathrm{Hilb}(W,t),$$

$$\mathrm{Hilb}(V,t) = \mathrm{Hilb}(V/W,t) + \mathrm{Hilb}(W,t),\ W\subset V.$$

Exercise 4.1.6 Prove that

$$\mathrm{Hilb}(K[x],t) = \frac{1}{1-t},\ \mathrm{Hilb}(K[x_1,\ldots,x_m],t_1,\ldots,t_m) = \prod_{i=1}^{m} \frac{1}{1-t_i},$$

$$\mathrm{Hilb}(K\langle x_1,\ldots,x_m\rangle,t_1,\ldots,t_m)=\frac{1}{1-(t_1+\ldots+t_m)},$$

$$\mathrm{Hilb}(K\langle x_1,\ldots,x_m\rangle,t)=\frac{1}{1-mt}.$$

Exercise 4.1.7 Consider the Grassmann algebra $E(V_m)$ of the m-dimensional vector space V_m with basis $\{e_1,\ldots e_m\}$ with the natural (multi)grading counting the entries of e_j in the monomials $e_{i_1}\ldots e_{i_n}$ (as in Exercise 4.1.2 (ii)). Show that

$$\mathrm{Hilb}(E(V_m),t_1,\ldots,t_m)=\sum_{j=0}^{m}e_j(t_1,\ldots,t_m)=\prod_{i=1}^{m}(1+t_i),$$

where $e_j(t_1,\ldots,t_m)$ is the j-th elementary symmetric polynomial.

4.2 Homogeneous and Multilinear Polynomial Identities

Now we go back to PI-algebras. Recall that the polynomial identity $g=0$ is a consequence of (or follows from) the polynomial identities $f_i=0$, $i\in I$, if $g\in\langle f_i\mid i\in I\rangle^T$, the T-ideal generated by f_i, $i\in I$.

Definition 4.2.1 Two sets of polynomial identities are *equivalent* if they generate the same T-ideal.

Definition 4.2.2 A polynomial $f(x_1,\ldots,x_n)$ in the free associative algebra $K\langle X\rangle$ (or in its graded subalgebras and factor algebras) is *multilinear of degree n* if f is multihomogeneous of degree $(1,\ldots,1)$ in $K\langle x_1,\ldots,x_n\rangle\subset K\langle X\rangle$. We denote by P_n the vector space of all polynomials in $K\langle X\rangle$ which are multilinear of degree n. (Very often one uses V_n instead of P_n, but we have reserved V_n for the n-dimensional vector space.) Clearly, P_n is of dimension $n!$ and has a basis

$$\{x_{\sigma(1)}\ldots x_{\sigma(n)}\mid\sigma\in S_n\}.$$

Proposition 4.2.3 *Let*

$$f(x_1,\ldots,x_m)=\sum_{i=0}^{n}f_i\in K\langle X\rangle,$$

where f_i is the homogeneous component of f of degree i in x_1.

(i) *If the base field K contains more that n elements (e.g. K is infinite), then the polynomial identities $f_i=0$, $i=0,1,\ldots,n$, follow from $f=0$.*

(ii) *If the base field is of characteristic* 0 *(or if* $\mathrm{char}K > \deg f$*), then* $f = 0$ *is equivalent to a set of multilinear polynomial identities.*

Proof. (i) Let $V = \langle f\rangle^T$ be the T-ideal of $K\langle X\rangle$ generated by f. We choose $n+1$ different elements $\alpha_0, \alpha_1, \ldots, \alpha_n$ of K. Since V is a T-ideal,

$$f(\alpha_j x_1, x_2, \ldots, x_m) = \sum_{i=0}^{n} \alpha_j^i f_i(x_1, x_2, \ldots, x_m) \in V,\ j = 0, 1, \ldots, n.$$

We consider these equations as a linear system with unknowns f_i, $i = 0, 1, \ldots, n$. Since its determinant

$$\begin{vmatrix} 1 & \alpha_0 & \alpha_0^2 & \ldots & \alpha_0^n \\ 1 & \alpha_1 & \alpha_1^2 & \ldots & \alpha_1^n \\ 1 & \alpha_2 & \alpha_2^2 & \ldots & \alpha_2^n \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \alpha_n & \alpha_n^2 & \ldots & \alpha_n^n \end{vmatrix} = \prod_{i<j}(\alpha_j - \alpha_i)$$

is the Vandermonde determinant and is different from 0, we obtain that each $f_i(x_1, \ldots, x_m)$ also belongs to V, i.e. the polynomial identities $f_i = 0$ are consequences of $f = 0$.

(ii) We use the *process of linearization.* By (i) we may assume that $f(x_1, \ldots, x_m)$ is homogeneous in each of its variables. Let $\deg_{x_1} f = d$. We write $f(y_1 + y_2, x_2, \ldots, x_m) \in V$ in the form

$$f(y_1 + y_2, x_2, \ldots, x_m) = \sum_{i=0}^{d} f_i(y_1, y_2, x_2, \ldots, x_m),$$

where f_i is the homogeneous component of degree i in y_1. Hence $f_i \in V$, $i = 0, 1, \ldots, d$. Since $\deg_{y_j} f_i < d$, $i = 1, \ldots, d-1$, $j = 1, 2$, we apply inductive arguments and obtain a set of multilinear consequences of $f = 0$. In order to see that these multilinear identities are equivalent to $f = 0$, it is sufficient to see that

$$f_i(y_1, y_1, x_2, \ldots, x_m) = \binom{d}{i} f(y_1, x_2, \ldots, x_m),$$

and the binomial coefficient is different from 0 because $\mathrm{char}K = 0$ or $\mathrm{char}K = p > d$.

Exercise 4.2.4 Let $\mathrm{char}K = 0$. Find a system of multilinear polynomial identities which is equivalent to each of the polynomial identities:

$$x_1x_2x_1x_2 + 2x_1x_2x_1^2 + x_2x_1x_3 = 0;$$

$$x_1^2 = 0;\ x_1^3 = 0;\ x_1^n = 0;$$

$$[x_1,x_2]^2=0.$$

Exercise 4.2.5 Show that any PI-algebra (over any field) satisfies a multilinear polynomial identity.

Hint. Starting with an arbitrary polynomial identity $f(x_1,\ldots,x_m)=0$, use that the degree in y_1 and the degree in y_2 of the consequence of $f=0$

$$g(y_1,y_2,x_2,\ldots,x_m)=f(y_1+y_2,x_2,\ldots,x_d)-$$
$$-f(y_1,x_2,\ldots,x_m)-f(y_2,x_2,\ldots,x_m)$$

are smaller than the degree of $f(x_1,\ldots,x_m)$ in x_1 and if $\deg_{x_1}f>1$, then $\deg_{y_j}g>0$, $j=1,2$.

In the case of characteristic 0 every T-ideal is generated by its multilinear polynomial identities. Traditionally, many of the results on PI-algebras are also stated in the language of multilinear identities.

Definition 4.2.6 Let R be a PI-algebra with T-ideal $T(R)$. The dimension of the multilinear polynomials in $K\langle X\rangle$ modulo the polynomial identities of R is called the *n-th codimension of the T-ideal $T(R)$ or of the polynomial identities of R* and is denoted by $c_n(R)$ (or by $c_n(\mathfrak{V})$ if we consider the T-ideal of the polynomial identities of a variety $\mathfrak{V}$), i.e.

$$c_n(R)=\dim P_n/(P_n\cap T(R)),\ n=0,1,2,\ldots$$

This sequence is called the *codimension sequence* of the T-ideal $T(R)$. We also consider the *codimension series* and the *exponential codimension series*

$$c(R,t)=\sum_{n\geq 0}c_n(R)t^n,\ \tilde{c}(R,t)=\sum_{n\geq 0}c_n(R)\frac{t^n}{n!}.$$

Convention 4.2.7 If R is an algebra (or $\mathfrak{V}$ is a variety of algebras), for any subset S of $K\langle X\rangle$ we denote by $S(R)$ (or by $S(\mathfrak{V})$) the image of S under the canonical homomorphisms

$$K\langle X\rangle\longrightarrow F(R)=F(\mathrm{var}R)=K\langle X\rangle/T(R),\ K\langle X\rangle\longrightarrow F(\mathfrak{V}).$$

For example, the formula for the codimension sequence of $T(R)$ is

$$c_n(R)=\dim P_n(R),\ n=0,1,2,\ldots$$

Definition 4.2.8 Let the T-ideal $T(R)$ be a (multi)homogeneous ideal of $K\langle X\rangle$. (For example this holds if the base field K is infinite.) Then we denote by

$$H(F_m(R), t_1, \dots, t_m) = \mathrm{Hilb}(F_m(R), t_1, \dots, t_m)$$

the Hilbert series of the relatively free algebra of rank m in $\mathrm{var}R$. If we are interested in the Hilbert series in one variable, we write

$$H(F_m(R), t) = \mathrm{Hilb}_m(\mathrm{var}R, t) = H_m(\mathrm{var}R, t) = \sum_{n \geq 0} \mathrm{dim}F_m^{(n)}(R)t^n.$$

4.3 Proper Polynomial Identities

The considerations in Section 4.2 hold for any class of algebras with the corresponding restrictions on the base field K. To the best of our knowledge, they were stated for the first time in the papers by A.I. Malcev [180] and Specht [245] in 1950. In what follows we shall use essentially that we work with unitary associative algebras. This investigation was started by Specht [245] in 1950 and in the quantitative way as here by Drensky [74], see also the expository paper [77]. The approach of [74] is based on an idea used by Volichenko [262] to describe the algebra $K\langle X\rangle/T(R)$ in the special case, when $\mathrm{char}K = 0$ and the T-ideal $T(R)$ is generated by $[x_1, x_2, x_3, x_4]$.

Definition 4.3.1 A polynomial $f \in K\langle X\rangle$ is called a *proper* (or *commutator*) polynomial, if it is a linear combination of products of commutators

$$f(x_1, \dots, x_m) = \sum \alpha_{i,\dots,j}[x_{i_1}, \dots, x_{i_p}] \dots [x_{j_1}, \dots, x_{j_q}],\ \alpha_{i,\dots,j} \in K.$$

(We assume that 1 is a product of an empty set of commutators.) We denote by B the set of all proper polynomials in $K\langle X\rangle$,

$$B_m = B \cap K\langle x_1, \dots, x_m\rangle,\ m = 1, 2, \dots,\ \Gamma_n = B \cap P_n,\ n = 0, 1, 2, \dots,$$

i.e. B_m is the set of the proper polynomials in m variables and Γ_n is the set of all proper multilinear polynomials of degree n. If R is a PI-algebra, by our Convention 4.2.7, we denote by $B(R)$, $B_m(R)$ and $\Gamma_n(R)$ the images in $F(R)$ of the corresponding vector subspaces of $K\langle X\rangle$.

Exercise 4.3.2 Let $B^{(n)}$ be the homogeneous component of degree n of B. Show that $B^{(0)} = K$, $B^{(1)} = 0$, and $B^{(2)}$ and $B^{(3)}$ have respectively bases

$$\{[x_i, x_j] \mid i > j\},\ \{[x_i, x_j, x_k] \mid i > j \leq k\}.$$

Proposition 4.3.3 (i) *Let us choose an ordered basis of the free Lie algebra* $L(X)$

$$x_1, x_2, \dots, [x_{i_1}, x_{i_2}], [x_{j_1}, x_{j_2}], \dots, [x_{k_1}, x_{k_2}, x_{k_3}], \dots,$$

consisting of the variables $x_1, x_2, \ldots$ *and some commutators, such that the variables precede the commutators. Then the vector space* $K\langle X\rangle$ *has a basis*

$$x_1^{a_1}\ldots x_m^{a_m}[x_{i_1},x_{i_2}]^b\ldots[x_{l_1},\ldots,x_{l_p}]^c,$$

where $a_1,\ldots,a_m,b,\ldots,c\geq 0$ *and* $[x_{i_1},x_{i_2}]<\ldots<[x_{l_1},\ldots,x_{l_p}]$ *in the ordering of the basis of* $L(X)$*. The basis elements of* $K\langle X\rangle$ *with* $a_1=\ldots=a_m=0$ *form a basis for the vector space* B *of the proper polynomials.*

(ii) *If* R *is a unitary PI-algebra over an infinite field* K*, then all polynomial identities of* R *follow from the proper ones (i.e. from those in* $T(R)\cap B$*). If* $\mathrm{char}K=0$*, then the polynomial identities of* R *follow from the proper multilinear identities (i.e. from those in* $T(R)\cap\Gamma_n$*,* $n=2,3,\ldots$*).*

Proof. (i) The first statement about the basis of $K\langle X\rangle$ follows from Witt Theorem 1.3.5 (that the free associative algebra $K\langle X\rangle$ is the universal enveloping algebra of the free Lie algebra $L(X)$) and from Poincaré-Birkhoff-Witt Theorem 1.3.2 which gives the basis of $U(G)$ for any Lie algebra G. The statement about the basis of B also follows from Poincaré-Birkhoff-Witt Theorem. If we express the product of commutators

$$[x_{i_1},\ldots,x_{i_p}]\ldots[x_{j_1},\ldots,x_{j_q}]$$

as a linear combination of the basis elements of $K\langle X\rangle$, the key point is that for any two consecutive commutators (both from the basis of $L(X)$) which are in a "wrong" order, e.g.

$$\ldots[x_{b_1},\ldots,x_{b_l}][x_{a_1},\ldots,x_{a_k}]\ldots,\ [x_{b_1},\ldots,x_{b_l}]>[x_{a_1},\ldots,x_{a_k}],$$

we replace the product

$$[x_{b_1},\ldots,x_{b_l}][x_{a_1},\ldots,x_{a_k}]$$

by the sum

$$[x_{a_1},\ldots,x_{a_k}][x_{b_1},\ldots,x_{b_l}]+[[x_{b_1},\ldots,x_{b_l}],[x_{a_1},\ldots,x_{a_k}]].$$

Since the second summand belongs to $L(X)$ and is a linear combination of commutators from the basis of $L(X)$, we can apply inductive arguments and see that the elements of B are linear combinations of

$$[x_{i_1},x_{i_2}]^b\ldots[x_{l_1},\ldots,x_{l_p}]^c,$$

as required.

(ii) Let $f(x_1,\ldots,x_m)=0$ be a polynomial identity for R. We may assume that f is homogeneous in each of its variables. We write f in the form

$$f=\sum\alpha_a x_1^{a_1}\ldots x_m^{a_m}w_a(x_1,\ldots,x_m),\ \alpha_a\in K,$$

where $w_a(x_1,\ldots,x_m)$ is a linear combination of

$$[x_{i_1}, x_{i_2}]^b \ldots [x_{l_1}, \ldots, x_{l_p}]^c.$$

Clearly, if we replace by 1 a variable in a commutator $[x_{i_1}, \ldots, x_{i_p}]$, the commutator vanishes. Since $f(1 + x_1, x_2, \ldots, x_m) = 0$ is also a polynomial identity for R, we obtain that

$$f(1+x_1, x_2, \ldots, x_m) = \sum \alpha_a \sum_{i=0}^{a_1} \binom{a_1}{i} x_1^i x_2^{a_2} \ldots x_m^{a_m} w_a(x_1, \ldots, x_m) \in T(R).$$

The homogeneous component of minimal degree with respect to x_1 is obtained from the summands with a_1 maximal among those with $\alpha_a \neq 0$. Since the T-ideal $T(R)$ is homogeneous, we obtain that

$$\sum_{a_1 \max} \alpha_a x_2^{a_2} \ldots x_m^{a_m} w_a(x_1, \ldots, x_m) \in T(R).$$

Multiplying from the left this polynomial identity by $x_1^{a_1}$ and subtracting the product from $f(x_1, \ldots, x_m)$, we obtain an identity which is similar to $f(x_1, \ldots, x_m)$ but involving lower values of a_1. By induction we establish that

$$\sum_{a_1 \text{ fixed}} \alpha_a x_2^{a_2} \ldots x_m^{a_m} w_a(x_1, \ldots, x_m) \in T(R),$$

$$w_a(x_1, \ldots, x_m) \in T(R),$$

and this completes the proof. The "multilinear" part of the statement is also clear. Starting with any multilinear polynomial identity for R, and doing exactly the same procedure as above, we obtain that the identity follows from some proper identities which are also multilinear.

Exercise 4.3.4 Express the polynomial

$$x_2 x_1^2 x_3 + 3 x_3 x_2 x_1^2 - x_1 x_3 x_2 x_1 \in K\langle X \rangle$$

as a linear combination of the basis vectors in Proposition 4.3.3.

Exercise 4.3.5 Let char$K = 0$. Find a system of proper multilinear polynomial identities, equivalent to the polynomial identity

$$x_2 x_1^2 x_3 - x_3 x_2 x_1^2 - x_1 x_3 x_2 x_1 + x_1 x_2 x_3 x_1 = 0.$$

Exercise 4.3.6 Let char$K = 0$. Show that a polynomial $f(x_1, \ldots, x_m) \in K\langle X \rangle$ is proper if and only if the formal partial derivatives $\partial f / \partial x_i$ are equal to 0 for $i = 1, \ldots, m$. Here $\partial / \partial x_i$ is the derivation of $K\langle X \rangle$ defined by

$$\frac{\partial x_j}{\partial x_i} = \delta_{ij} \text{ (the Kronecker symbol).}$$

In virtue of this property, sometimes one calls the elements of Γ_n *multilinear constants.*

Exercise 4.3.7 Let R be any commutative algebra over an infinite field K. Show that all polynomial identities of R follow from $[x_1, x_2] = 0$.

Exercise 4.3.8 Let $L(X)$ be the free Lie algebra considered as a subalgebra of $K\langle X\rangle^{(-)}$ and let $PL_n = P_n \cap L(X)$ be the set of the multilinear Lie polynomials. Prove that for $n > 1$, the vector space PL_n has a basis

$$\{[x_n, x_{\sigma(1)}, \ldots, x_{\sigma(n-1)}] \mid \sigma \in S_{n-1}\}.$$

Hint. Using the Jacobi identity and the anticommutative law, show that these elements span the vector space PL_n. Write $[x_n, x_{\sigma(1)}, \ldots, x_{\sigma(n-1)}]$ as a linear combination of monomials $x_{\tau(1)} \ldots x_{\tau(n)}$, $\tau \in S_n$. Show that $x_n x_{\sigma(1)} \ldots x_{\sigma(n-1)}$ is the leading term of $[x_n, x_{\sigma(1)}, \ldots, x_{\sigma(n-1)}]$ with respect to the lexicographic ordering in $K\langle X\rangle$.

The basis of Γ_n given in the next theorem is called the *Specht basis.*

Theorem 4.3.9 *A basis of the vector space Γ_n of all proper multilinear polynomials of degree $n \geq 2$ consists of the following products of commutators*

$$[x_{i_1}, \ldots, x_{i_k}] \ldots [x_{j_1}, \ldots, x_{j_l}],$$

where:

(i) *All products are multilinear in the variables $x_1, \ldots, x_n$;*

(ii) *Each factor $[x_{p_1}, x_{p_2}, \ldots, x_{p_s}]$ is a left normed commutator of length ≥ 2 and the maximal index is in the first position, i.e. $p_1 > p_2, \ldots, p_s$;*

(iii) *In each product the shorter commutators precede the longer, i.e. in the beginning of the statement of the theorem $k \leq \ldots \leq l$;*

(iv) *If two consecutive factors are commutators of equal length, then the first variable of the first commutator is smaller that the first variable in the second one, i.e.*

$$\ldots [x_{p_1}, x_{p_2}, \ldots, x_{p_s}][x_{q_1}, x_{q_2}, \ldots, x_{q_s}] \ldots$$

satisfies $p_1 < q_1$.

Proof. We choose a basis of $L(X)$ as in Proposition 4.3.3 (i) assuming that in the ordering of the basis the shorter commutators are before the longer ones and if two commutators are of the same length, this with a smaller first variable is before the other. Now the theorem follows directly from Proposition 4.3.3 (i) and Exercise 4.3.8.

Definition 4.3.10 Let R be a (unitary) PI-algebra over a field of characteristic 0. By analogy with Definition 4.2.6 we introduce the vector subspace $\Gamma_n(R) = \Gamma_n/(\Gamma_n \cap T(R))$, the *proper codimension sequence*

$$\gamma_n(R) = \dim \Gamma_n(R),\ n = 0, 1, 2, \ldots,$$

and the *proper codimension series* and *exponential proper codimension series*

$$\gamma(R,t) = \sum_{n \geq 0} \gamma_n(R) t^n,\ \tilde{\gamma}(R,t) = \sum_{n \geq 0} \gamma_n(R) \frac{t^n}{n!}.$$

Now we give some quantitative relations between the ordinary and the proper polynomial identities.

Theorem 4.3.11 *Let R be a unitary PI-algebra over an infinite field K.*

(i) *Let*

$$\{w_j(x_1, \ldots, x_m) \mid j = 1, 2, \ldots\}$$

be a basis of the vector space $B_m(R)$ of the proper polynomials in the relatively free algebra $F_m(R)$ of rank m, i.e.

$$B_m(R) = (K\langle x_1, \ldots, x_m\rangle \cap B)/(T(R) \cap K\langle x_1, \ldots, x_m\rangle \cap B).$$

Then $F_m(R)$ has a basis

$$\{x_1^{a_1} \ldots x_m^{a_m} w_j(x_1, \ldots, x_m) \mid a_i \geq 0,\ j = 0, 1, \ldots\}.$$

(ii) *If*

$$\{u_{jk}(x_1, \ldots, x_k) \mid j = 1, 2, \ldots, \gamma_k(R)\}$$

is a basis of the vector space $\Gamma_k(R)$ of the proper multilinear polynomials of degree k in $F(R)$, then $P_n(R)$ has a basis consisting of all multilinear polynomials of the form

$$x_{p_1} \ldots x_{p_{n-k}} u_{jk}(x_{q_1}, \ldots, x_{q_k}),\ j = 1, 2, \ldots, \gamma_k(R), k = 0, 1, 2, \ldots, n,$$

such that $p_1 < \ldots < p_{n-k}$ and $q_1 < \ldots < q_k$.

Proof. (i) Let $w'_j(x_1, \ldots, x_m) \in B_m$ be homogeneous preimages of the elements $w_j(x_1, \ldots, x_m) \in B_m(R)$, $j = 1, 2, \ldots$ We choose an arbitrary homogeneous basis $\{v_k \mid k = 1, 2, \ldots\}$ of $B_m \cap T(R)$. Then

$$\{w'_j(x_1, \ldots, x_m), v_k \mid j = 1, 2, \ldots, k = 1, 2, \ldots\}$$

is a homogeneous basis of B_m. Applying Proposition 4.3.3, we see that $F_m(R)$ is spanned by

$$x_1^{a_1} \ldots x_m^{a_m} w_j(x_1, \ldots, x_m),\ a_i \geq 0, j = 1, 2, \ldots$$

and these elements are linearly independent. The proof of (ii) is similar.

Theorem 4.3.12 *Let R be a unitary PI-algebra over an infinite field K.*

(i) *The Hilbert series of the relatively free algebra $F_m(R)$ and its proper elements $B_m(R)$ are related by*

$$\mathrm{Hilb}(F_m(R), t_1, \ldots, t_m) = \mathrm{Hilb}(B_m(R), t_1, \ldots, t_m) \prod_{i=1}^{m} \frac{1}{1-t_i},$$

$$\mathrm{Hilb}(F_m(R), t) = \mathrm{Hilb}(B_m(R), t) \frac{1}{(1-t)^m}.$$

(ii) *The codimensions and the proper codimensions of the polynomial identities of R and the corresponding formal power series are related as follows:*

$$c_n(R) = \sum_{k=0}^{n} \binom{n}{k} \gamma_k(R), \ n = 0, 1, 2, \ldots,$$

$$c(R, t) = \frac{1}{1-t} \gamma(R, \frac{t}{1-t}),$$

$$\tilde{c}(R, t) = \exp(t) \tilde{\gamma}(R, t).$$

Proof. The assertion of (i) follows immediately from Theorem 4.3.11 and Exercise 4.1.6. The first statement of (ii) also is a consequence of Theorem 4.3.11 and the equations involving formal power series can be obtained by easy manipulation of the first equation of (ii).

5. Polynomial Identities of Concrete Algebras

Now we shall apply the results of Chapter 4 to study the polynomial identities of the Grassmann algebra and of the algebra of $k \times k$ upper triangular matrices. Using other methods, the polynomial identities of the Grassmann algebra were described by Krakowski and Regev [155], together with some additional information. For further development see the papers by Berele and Regev [32] and Luisa Carini [45], e.g. for the Hilbert series of the corresponding relatively free algebra. Using methods similar to ours, another proof was given by Di Vincenzo [63]. The basis of the polynomial identities for the upper triangular matrices was found by Yu.N. Maltsev [181] in the case of characteristic 0 and by several other authors (e.g. Siderov [243], Kalyulaid [131], Polin [208], see the book by Vovsi [265]) in the case of any field. See also the book by Bahturin [21] for the Lie algebra case. A lot of information on the T-ideals, containing all polynomial identities of the algebra of $k \times k$ upper triangular matrices can be found in the paper by Latyshev [165], see also the paper by Drensky [76]. Apart from the Grassmann algebra and the upper triangular matrix algebras, over a field of characteristic 0, the bases of the polynomial identities of concrete algebras are known in few cases only. The most interesting algebras among them are the algebra of 2×2 matrices (Razmyslov [217] in 1973, and a minimal basis of the identities, Drensky [72] in 1981), the tensor square of the Grassmann algebra $E \otimes E$ (Popov [209] in 1982). Over an infinite field of positive characteristic the known results are even less. Over a finite field one uses completely another technique and the best known results contain the bases of the polynomial identities for $M_k(\mathbb{F}_q)$ for $k = 2, 3, 4$, respectively due to Yu.N. Maltsev and Kuzmin [182], Genov [111] and Genov and Siderov [113] in 1978-1982. We refer also to the survey article [77] for other applications of the methods of our exposition.

5.1 Polynomial Identities of the Grassmann Algebra

In this section we shall describe the polynomial identities of the Grassmann algebra and their numerical invariants. We assume that K is a fixed field of characteristic 0.

Lemma 5.1.1 *Let* $G = \langle [x_1, x_2, x_3] \rangle^T$ *be the T-ideal of* $K\langle X\rangle$ *generated by the polynomial identity* $[x_1, x_2, x_3]$. *Then the polynomials*

$$[x_1, x_2][x_2, x_3] \text{ and } [x_1, x_2][x_3, x_4] + [x_1, x_3][x_2, x_4]$$

belong to G.

Proof. We apply the identity

$$[uv, w] = [u, w]v + u[v, w]$$

(which expresses the fact that the commuting with a fixed element is a derivation) and obtain from $[x_1, x_2^2, x_3] \in G$ that

$$[x_1, x_2^2, x_3] = [[x_1, x_2]x_2 + x_2[x_1, x_2], x_3] =$$

$$= [x_1, x_2, x_3]x_2 + [x_1, x_2][x_2, x_3] + [x_2, x_3][x_1, x_2] + x_2[x_1, x_2, x_3] \in G,$$

$$[x_1, x_2][x_2, x_3] + [x_2, x_3][x_1, x_2] \in G,$$

$$[x_1, x_2][x_2, x_3] + [x_2, x_3][x_1, x_2] = 2[x_1, x_2][x_2, x_3] + [[x_2, x_3], [x_1, x_2]] \in G$$

and, since $[[x_2, x_3], [x_1, x_2]] \in G$, this gives that

$$[x_1, x_2][x_2, x_3] \in G.$$

The linearization of this polynomial is

$$[x_1, x_2][x_2', x_3] + [x_1, x_2'][x_2, x_3]$$

(which is the same as $[x_1, x_2][x_3, x_4] + [x_1, x_3][x_2, x_4]$) and also belongs to G.

Theorem 5.1.2 *Let* $\mathrm{char} K = 0$ *and let* E *be the Grassmann algebra of an infinite dimensional vector space.*

(i) *The T-ideal* $T(E)$ *is generated by* $[x_1, x_2, x_3]$.

(ii) *The codimensions of the identities of* E *satisfy*

$$c_n(E) = 2^{n-1},\ n = 1, 2, \ldots,$$

$$c(E, t) = \sum_{n\geq 0} c_n(E)t^n = \frac{1+t}{1-2t},$$

$$\tilde{c}(E, t) = \sum_{n\geq 0} c_n(E)\frac{t^n}{n!} = \frac{1}{2}(1 + e^{2t}).$$

(iii) *The Hilbert series of the relatively free algebra* $F_m(E)$ *are*

$$\mathrm{Hilb}(F_m(E), t_1, \ldots, t_m) = \frac{1}{2} + \frac{1}{2}\prod_{i=1}^{m}\frac{1+t_i}{1-t_i},$$

$$\mathrm{Hilb}(F_m(E), t) = \frac{1}{2} + \frac{1}{2}\left(\frac{1+t}{1-t}\right)^m .$$

Proof. We shall apply the results of Chapter 4. Recall that B and Γ_n are respectively the sets of all proper polynomials in $K\langle X\rangle$ and of all proper multilinear polynomials of degree n. Every T-ideal is generated by its proper elements.

(i) Since the Grassmann algebra satisfies the identity $[x_1, x_2, x_3] = 0$ (see Exercise 2.1.4), we obtain that $\langle [x_1, x_2, x_3]\rangle^T \subseteq T(E)$. Let

$$w = [x_{i_1}, \ldots, x_{i_k}] \ldots [x_{j_1}, \ldots, x_{j_l}] \in K\langle X\rangle$$

be any product of commutators. Hence, if $w \notin \langle [x_1, x_2, x_3]\rangle^T$, then

$$w = [x_{i_1}, x_{i_2}] \ldots [x_{i_{2k-1}}, x_{i_{2k}}].$$

The anticommutative law $[x_i, x_j] = -[x_j, x_i]$ and Lemma 5.1.1 give that, modulo the T-ideal $\langle [x_1, x_2, x_3]\rangle^T$, we can change the places of the variables in w. In particular, if two of x_{i_p} and x_{i_q} in w are equal, we obtain that $w \in \langle [x_1, x_2, x_3]\rangle^T$. Hence $B/(B \cap \langle [x_1, x_2, x_3]\rangle^T)$ is spanned by

$$w = [x_{i_1}, x_{i_2}] \ldots [x_{i_{2k-1}}, x_{i_{2k}}],\ i_1 < i_2 < \ldots < i_{2k-1} < i_{2k},\ k = 0, 1, \ldots$$

We shall prove the statement if we establish that any nontrivial linear combination of these elements is different from 0 in E. Since w is uniquely determined by its multidegree and $T(E)$ is homogeneous, we have to prove that w itself is not a polynomial identity for E. But

$$[e_1, e_2] \ldots [e_{2k-1}, e_{2k}] = 2^k e_1 e_2 \ldots e_{2k-1} e_{2k} \neq 0$$

in E and this completes the proof.

(ii) From the proof of (i) we obtain that $\Gamma_n(E)$ is spanned on

$$[x_1, x_2] \ldots [x_{2k-1}, x_{2k}],\ n = 2k,$$

and $\Gamma_n(E) = 0$ if n is odd. Hence, if $\varepsilon_n = 1$ for n even and $\varepsilon_n = 0$ for n odd, then

$$\gamma_n(E) = \dim \Gamma_n(E) = \varepsilon_n = \frac{1}{2}(1 + (-1)^n),\ n = 0, 1, 2, \ldots,$$

$$\gamma(E, t) = \sum_{k \geq 0} t^{2k} = \frac{1}{1 - t^2},$$

$$\tilde{\gamma}(E,t) = \frac{1}{2}(e^t + e^{-t}),$$

and the proof follows from Theorem 4.3.12 (ii) by easy calculations.

(iii) As in (ii), $B_m(E)$ has a basis

$$\{[x_{i_1}, x_{i_2}]\dots[x_{i_{2k-1}}, x_{i_{2k}}] \mid 1 \le i_1 < i_2 < \dots < i_{2k-1} < i_{2k} \le m\}.$$

The Hilbert series of $B_m(E)$ is equal to the sum of the elementary symmetric polynomials

$$\sum_{k\ge 0} e_{2k}(t_1,\dots,t_m) = \frac{1}{2}\left(\prod_{i=1}^{m}(1-t_i) + \prod_{i=1}^{m}(1+t_i)\right)$$

and we apply Theorem 4.3.12 (i).

Exercise 5.1.3 Show that the polynomial identities of the Grassmann algebra E_k of the k-dimensional vector space V_k, $k > 1$, follow from $[x_1, x_2, x_3] = 0$ and the standard identity $s_{2p}(x_1,\dots,x_{2p}) = 0$, where p is the minimal integer with $2p > k$.

Hint. Show that $s_{2p}(x_1,\dots,x_{2p}) = 0$ is not a polynomial identity for $E(V_{2p})$ and, modulo the T-ideal generated by $[x_1, x_2, x_3]$, the identity

$$[x_1, x_2]\dots[x_{2p+1}, x_{2p+2}] = 0,$$

which holds for $E(V_{2p+1})$, is equivalent to the standard identity $s_{2p+2} = 0$.

5.2 Polynomial Identities of the Upper Triangular Matrices

Theorem 5.2.1 *Let K be any infinite field and let $U_k(K)$ be the algebra of $k \times k$ upper triangular matrices.*

(i) *The polynomial identity*

$$[x_1, x_2]\dots[x_{2k-1}, x_{2k}] = 0$$

forms a basis of the polynomial identities of $U_k(K)$.

(ii) *The relatively free algebra $F(U_k(K))$ has a basis consisting of all products*

$$x_1^{a_1}\dots x_m^{a_m}[x_{i_{11}}, x_{i_{21}},\dots, x_{i_{p_1 1}}]\dots[x_{i_{1r}}, x_{i_{2r}},\dots, x_{i_{p_r r}}],$$

where the number r of participating commutators is $\le k-1$ and the indices in each commutator $[x_{i_{1s}}, x_{i_{2s}},\dots, x_{i_{p_s s}}]$ satisfy $i_{1s} > i_{2s} \le \dots \le i_{p_s s}$.

Proof. We have seen in Exercise 2.1.9 that $U_k(K)$ satisfies the polynomial identity

$$[x_1, x_2] \dots [x_{2k-1}, x_{2k}] = 0$$

and we shall show that all other polynomial identities for $U_k(K)$ follow from this identity. We use the same idea as in the proof for the Grassmann algebra in Theorem 5.1.2. We shall see that modulo this identity every element of $K\langle X\rangle$ is a linear combination of the elements from part (ii) of the theorem and that any nontrivial linear combination of these elements does not vanish on the algebra $U_k(K)$. Following our philosophy, we shall consider proper polynomial identities only. It is also convenient to work in the relatively free algebra

$$F(\mathfrak{T}_k) = K\langle X\rangle / \langle [x_1, x_2] \dots [x_{2k-1}, x_{2k}]\rangle^T,$$

where, in this section only, we denote by $\mathfrak{T}_k$ the variety defined by the polynomial identity under consideration. In other words, we work modulo the T-ideal

$$T(\mathfrak{T}_k) = \langle [x_1, x_2] \dots [x_{2k-1}, x_{2k}]\rangle^T.$$

For simplicity of the exposition we give the proof for the cases $k = 2$ and $k = 3$ only. The general case is similar.

Let $k = 2$. Then in $F(\mathfrak{T}_2)$

$$[x_1, x_2][x_3, x_4] = 0$$

and $B(\mathfrak{T}_2)$ is spanned by 1 and by all commutators

$$[x_{i_1}, x_{i_2}, \dots, x_{i_n}],\ n \geq 2.$$

Using the identity

$$0 = [x_1, x_2][x_3, x_4] - [x_3, x_4][x_1, x_2] = [[x_1, x_2], [x_3, x_4]] =$$

$$= [x_1, x_2, x_3, x_4] - [x_1, x_2, x_4, x_3],$$

we see that in $F(\mathfrak{T}_2)$

$$[y_1, y_2, x_{\sigma(1)}, \dots, x_{\sigma(p)}] = [y_1, y_2, x_1, \dots, x_p],\ \sigma \in S_p.$$

Additionally, the Jacobi identity and the anticommutativity allow to change the places of the variables in the first three positions:

$$[x_1, x_2] = -[x_2, x_1],\ [x_3, x_2, x_1] = [x_3, x_1, x_2] - [x_2, x_1, x_3].$$

In this way, we can assume that $B(\mathfrak{T}_2)$ is spanned by 1 and

$$[x_{i_1}, x_{i_2}, x_{i_3}, \dots, x_{i_n}],\ i_1 > i_2 \leq i_3 \leq \dots \leq i_n.$$

We shall show that these elements are linearly independent modulo the T-ideal of $U_2(K)$. Let

$$f(x_1, \dots, x_m) = \sum_i \alpha_i [x_{i_1}, x_{i_2}, \dots, x_{i_n}] = 0,\ i_1 > i_2 \leq \dots \leq i_n,\ \alpha_i \in K,$$

be a nontrivial polynomial identity for $U_2(K)$. We fix i_1 maximal with the property $\alpha_i \neq 0$ and consider the elements

$$\bar{x}_{i_1} = e_{12} + \xi_{i_1} e_{22},\ \bar{x}_j = \xi_j e_{22},\ j \neq i_1,\ \xi_{i_1}, \xi_j \in K.$$

Concrete calculations show that

$$f(\bar{x}_1, \ldots, \bar{x}_m) = \left(\sum_{i_1 \text{fixed}} \alpha_i \xi_{i_2} \ldots \xi_{i_n} \right) e_{12}$$

which can be chosen different from 0 because the base field K is infinite. Hence all coefficients α_i are equal to 0 and this completes the proof for $k = 2$.

Now, let $k = 3$. Then in $F(\mathfrak{T}_3)$

$$[x_1, x_2][x_3, x_4][x_5, x_6] = 0$$

and $B(\mathfrak{T}_3)$ is spanned by 1, all commutators

$$[x_{i_1}, x_{i_2}, \ldots, x_{i_n}],\ n \geq 2.$$

and all products of two commutators

$$[x_{i_1}, x_{i_2}, \ldots, x_{i_p}][x_{j_1}, x_{j_2}, \ldots, x_{j_q}].$$

Applying the identity

$$[[y_1, y_2], [y_3, y_4], y_5] = [[y_1, y_2, y_5], [y_3, y_4]] + [[y_1, y_2], [y_3, y_4, y_5]],$$

we see that $[[y_1, \ldots, y_a], [z_1, \ldots, z_b], t_1, \ldots, t_c]$ is a linear combination of products of two commutators. As in the case $k = 2$ we see that $B(\mathfrak{T}_3)$ is spanned by 1,

$$[x_{i_1}, x_{i_2}, \ldots, x_{i_n}],\ i_1 > i_2 \leq \ldots \leq i_n,$$

$$[x_{i_1}, x_{i_2}, \ldots, x_{i_p}][x_{j_1}, x_{j_2}, \ldots, x_{j_q}],\ i_1 > i_2 \leq \ldots \leq i_p,\ j_1 > j_2 \leq \ldots \leq j_q,$$

and it is sufficient to show that these elements are linearly independent. Considering a linear combination and replacing $x_1, x_2, \ldots$ by 2×2 upper triangular matrices (regarded as 3×3 upper triangular matrices with zero entries in the third row and in the third column) we may assume that

$$f(x_1, \ldots, x_m) = \sum \alpha_{ij} [x_{i_1}, x_{i_2}, \ldots, x_{i_p}][x_{j_1}, x_{j_2}, \ldots, x_{j_q}] = 0,\ \alpha_{ij} \in K.$$

Now we consider a maximal pair (i_1, j_1) with $\alpha_{ij} \neq 0$ (first maximal in i_1 and between all such pairs, maximal in j_1). Let

$$\bar{x}_{i_1} = e_{12} + \xi_{i_1} e_{22} + \eta_{i_1} e_{33},\ \bar{x}_{j_1} = e_{23} + \xi_{j_1} e_{22} + \eta_{j_1} e_{33},$$

$$\bar{x}_l = \xi_l e_{22} + \eta_l e_{33},\ l \neq i_1, j_1,\ \xi_{i_1}, \xi_{j_1}, \xi_l, \eta_{i_1}, \eta_{j_1}, \eta_l \in K.$$

If $i_1 = j_1$, then we assume that

$$\bar{x}_{i_1} = e_{12} + e_{23} + \xi_{i_1}e_{22} + \eta_{i_1}e_{33}.$$

Again,

$$f(\bar{x}_1, \ldots, \bar{x}_m) = \left(\sum \alpha_{ij}\xi_{i_2}\ldots\xi_{i_p}(\eta_{j_2} - \xi_{j_2})\ldots(\eta_{j_q} - \xi_{j_q})\right) e_{13}$$

and this can be made different from 0. Therefore the above products of commutators are linearly independent modulo the polynomial identities of $U_3(K)$ and this completes the proof of the case $k = 3$.

Remark 5.2.2 Let R be a finitely generated PI-algebra satisfying a *nonmatrix polynomial identity* (i.e. an identity which does not hold for the 2×2 matrix algebra $M_2(K)$), the field K being infinite. A result of Latyshev [162] from 1966 gives that R satisfies some polynomial identity of the form

$$[x_1, x_2] \ldots [x_{2k-1}, x_{2k}] = 0.$$

From this point of view the polynomial identities of the upper triangular matrices serve as a measure how complicated are the polynomial identities of R. Nowadays the theorem of Latyshev is a direct consequence of the classical theorem of Amitsur that over an infinite field the T-ideals of the matrix algebras are the only prime ideals (see the proof e.g. in the book by Rowen [231]) and of the more recent theorem of Razmyslov-Kemer-Braun [220, 140, 38] that the Jacobson radical of every finitely generated PI-algebra is nilpotent.

Exercise 5.2.3 Let L_m/L''_m and L/L'' be respectively the free metabelian Lie algebra of rank m and of countable rank over an infinite field. Let $U_2(K)^{(-)}$ be the Lie algebra of 2×2 upper triangular matrices.

(i) Show that L/L'' has a basis

$$\{x_i, [x_{i_1}, x_{i_2}, \ldots, x_{i_n}] \mid i = 1, 2, \ldots, \; i_1 > i_2 \leq i_3 \leq \ldots \leq i_n, \; n \geq 2\}.$$

(ii) Show that the metabelian polynomial identity

$$[[x_1, x_2], [x_3, x_4]] = 0$$

forms a basis for the identities of the algebra $U_2(K)^{(-)}$.

(iii) Calculate the codimensions of the identities of $U_2(K)^{(-)}$. Calculate the ordinary and the exponential codimension series.

(iv) Calculate the Hilbert series of L_m/L''_m.

Hints. (i), (ii) Repeat the arguments of the proof in the associative case.
(iii) Answer:

$$c_1(U_2(K)^{(-)}) = 1, \; c_n(U_2(K)^{(-)}) = n - 1, \; n > 1,$$

$$\tilde{c}(U_2(K)^{(-)}, t) = t + (t - 1)e^t + 1.$$

(iv) Consider the vector subspace W of $K\langle x_1, \ldots, x_m\rangle$ with basis

$$\{x_{i_1}(x_{i_2}\dots x_{i_n}) \mid i_2\leq\dots\leq i_n, n\geq 2\}.$$

Its Hilbert series is

$$(t_1+\dots+t_m)\left(\mathrm{Hilb}(K[x_1,\dots,x_m],t_1,\dots,t_m)-1\right)=$$

$$=(t_1+\dots+t_m)\left(\prod_{i=1}^{m}\frac{1}{1-t_i}-1\right).$$

Let W_1 be the vector space of the polynomials without constant and linear terms and let $W_2=L'_m/L''_m$ be the commutator ideal of L_m/L''_m. Define a linear mapping $\phi: W\to W'\oplus W''$ by

$$\phi(x_{i_1}x_{i_2}\dots x_{i_n})=x_{i_1}x_{i_2}\dots x_{i_n}\in W_1,\ \text{if } i_1\leq i_2,$$

$$\phi(x_{i_1}x_{i_2}\dots x_{i_n})=[x_{i_1},x_{i_2},\dots,x_{i_n}]\in W_2,\ \text{if } i_1> i_2.$$

Show that ϕ is an isomorphism of graded vector spaces. Derive from here that

$$\mathrm{Hilb}(L_m/L''_m,t_1,\dots,t_m)=\sum_{i=1}^{m}t_i+\left(\sum_{i=1}^{m}t_i-1\right)\prod_{i=1}^{m}\frac{1}{1-t_i}.$$

Exercise 5.2.4 Let G be the two-dimensional nonnilpotent Lie algebra over a field of characteristic 0. (If G has a basis $\{a,b\}$, then, up to an isomorphism, the multiplication in G is given by $[a,b]=a$.) Show that G generates the metabelian variety of Lie algebras, i.e. the T-ideal of G is generated by the metabelian identity.

Hint. Show that G satisfies the metabelian polynomial identity and the $n-1$ multilinear elements of degree $n>1$ in Exercise 5.2.3 (i) are linearly independent in $F(G)$.

Exercise 5.2.5 Let R be the algebra of the 3×3 upper triangular matrices of the form

$$\begin{pmatrix}\lambda & * & *\\ 0 & \lambda & *\\ 0 & 0 & \lambda\end{pmatrix},$$

where the $*$'s denote arbitrary elements of the field, and let $\mathrm{char}K=0$. Show that

$$T(R)=\langle[x_1,x_2,x_3],s_4(x_1,x_2,x_3,x_4)\rangle^T.$$

Is this algebra isomorphic to the Grassmann algebra of the two-dimensional vector space?

Hint. Show that the algebra satisfies the given polynomial identities and use the scheme of the proof for the polynomial identities of E in Theorem 5.1.2.

A possible way to show that the algebras R and $E(V_2)$ are not isomorphic is the following. The vector subspaces of R and $E(V_2)$ spanned respectively by $\{e_{12}, e_{13}, e_{23}\}$ and $\{e_1, e_2, e_1e_2\}$ are the only maximal nilpotent ideals of these algebras. In the ideal of $E(V_2)$ the square of every element is 0 and in the ideal of R there exist elements which do not share this property.

Exercise 5.2.6 Let

$$\tilde{a}(t) = \sum_{p\geq 0} a_p\frac{t^p}{p!},\ \tilde{b}(t) = \sum_{q\geq 0} b_q\frac{t^q}{q!}.$$

Show that

$$\tilde{a}(t)\tilde{b}(t) = \sum_{n\geq 0}\left(\sum_{k=0}^{n}\binom{n}{k}a_kb_{n-k}\right)\frac{t^n}{n!}.$$

Exercise 5.2.7 Show that

$$\tilde{c}(U_k(K),t) = e^t\sum_{p=0}^{k-1}((t-1)e^t+1)^p = \frac{1}{1-t}(1-((t-1)e^t+1)^k).$$

Hint. Use the description of the multilinear elements of the relatively free algebra $F(U_k(K))$ in Theorem 5.2.1 (ii), Exercise 5.2.3 (iii) and Exercise 5.2.6.

Exercise 5.2.8 Find a basis of the polynomial identities for the Lie algebra of $k\times k$ upper triangular matrices over an infinite field. For $k = 3, 4$ find the codimensions of the T-ideal and the Hilbert series of the relatively free algebra.

Hint. Try to find the basis of the identities

$$[[x_1,x_2],[x_3,x_4],[x_5,x_6]] = 0,\ [[x_1,x_2],[x_3,x_4],[x_5,x_6],[x_7,x_8]] = 0,$$

respectively for $k = 3$ and $k = 4$ without consulting books. For the basis of the identities for $k > 4$ see e.g. the book by Bahturin [21]. For the Hilbert series and the codimensions try to modify the solutions of the previous exercises.

6. Methods of Commutative Algebra

Many results on PI-algebras generalize well known results of commutative algebra. On the other hand, methods of commutative algebra are very effective in the study of PI-algebras. In this chapter we give a background on commutative algebra including the Hilbert-Serre theorem for rationality of Hilbert series of finitely generated graded modules of polynomial algebras in several variables. We apply this theorem to varieties of algebras satisfying nonmatrix polynomial identities. We also give some idea of commutative and noncommutative invariant theory. There are many well written books on different aspects of invariant theory and we have limited our considerations to a sequence of exercises and remarks, with references to books and expository articles.

6.1 Rational Hilbert Series

Definition 6.1.1 The *generating function* of the sequence $a_0, a_1, a_2, \dots$ of elements of K is defined as the formal power series

$$a(t) = \sum_{n\geq 0} a_n t^n.$$

Exercise 6.1.2 Let f_n, $n = 0, 1, 2, \dots$, be the sequence of the *Fibonacci numbers* defined by

$$f_0 = f_1 = 1,\ f_{n+2} = f_{n+1} + f_n,\ n = 0, 1, 2, \dots$$

(i) Show that its generating function is equal to

$$f(t) = \sum_{n\geq 0} f_n t^n = \frac{1}{1-t-t^2}.$$

(ii) Prove that $f_n = c_1\alpha_1^n + c_2\alpha_2^n$, where c_1, c_2 are fixed numbers and α_1, α_2 are solutions of the quadratic equation $\xi^2 = \xi + 1$.

Hint. (i) Show that

$$f(t) = tf(t) + t^2 f(t) + f_0 + (f_1 - f_0)t.$$

(ii) Prove that all sequences $a_0, a_1, a_2, \ldots$, satisfying $a_{n+2} = a_{n+1} + a_n$, $n = 0, 1, 2, \ldots$, form a two-dimensional vector space with basis consisting of the two sequences

$$1, \alpha_i, \alpha_i^2, \ldots, \; i = 1, 2.$$

Exercise 6.1.3 Let $a_0, a_1, a_2, \ldots$ be a sequence and let $a(t) = \sum_{n \geq 0} a_n t^n$ be its generating function.

(i) Show that the sequence satisfies a *linear recurrence relation*

$$a_{n+p} = \beta_1 a_{n+p-1} + \ldots + \beta_{p-1} a_{n+1} + \beta_p a_n, \; n = 0, 1, 2 \ldots,$$

(for some fixed numbers $\beta_1, \ldots, \beta_p$) if and only if the generating function $a(t)$ is rational.

(ii) Let the sequence satisfy the above recurrence relation and let the different zeros of the equation

$$\xi^p = \beta_1 \xi^{p-1} + \ldots + \beta_{p-2}\xi^2 + \beta_{p-1}\xi + \beta_p,$$

be $\alpha_1, \ldots, \alpha_k$ with multiplicities respectively equal to $d_1, \ldots, d_k$. Show that

$$a_n = c_1(n)\alpha_1^n + \ldots + c_k(n)\alpha_k^n, \; n = 0, 1, 2, \ldots,$$

where each $c_i(n)$ is a polynomial in n and $\deg c_i(n) \leq d_i - 1$, $i = 1, \ldots, k$.

Hint. Repeat the arguments of Exercise 6.1.2.

Exercise 6.1.4 If $f(t) \in \mathbb{C}[t]$ is a polynomial such that $f(\mathbb{Z}) \subset \mathbb{Z}$, prove that

$$f(t) = \sum_{k=0}^{p} \frac{a_k}{k!} t(t-1)\ldots(t-(k-1)), \; a_k \in \mathbb{Z}.$$

Recall that a vector space M is a *(left) module* of the algebra A if an algebra homomorphism $A \longrightarrow \mathrm{End}_K(M)$ is given. (Since we work with unitary algebras, we assume that the unity of A maps to the identity linear operator on M.) The commutative algebra is *noetherian* if its ideals satisfy the *ascending chain condition*: For every infinite sequence of ideals

$$I_1 \subseteq I_2 \subseteq I_3 \subseteq \ldots$$

there exists an integer n such that

$$I_n = I_{n+1} = I_{n+2} = \ldots$$

Hilbert Basis Theorem 6.1.5 *If the commutative algebra A is noetherian, then the polynomial algebra $A[x]$ is also noetherian.*

Exercise 6.1.6 (i) Find in the books (e.g. in the book by Lang [161] or Atyiah and Macdonald [16]) and read the proof of Hilbert Basis Theorem 6.1.5.

(ii) Prove that any finitely generated commutative algebra is noetherian.

(iii) If M is a finitely generated module of a finitely generated commutative algebra A, then every A-submodule of M is also finitely generated.

Exercise 6.1.7 Let $A = K[x_1, \dots, x_m]$ and let S be the subalgebra of all symmetric polynomials in A. Show that A is a finitely generated S-module.

Hint. Let $e_i = e_i(x_1, \dots, x_m)$ be the i-th elementary symmetric polynomial. Then for a fixed k,

$$\prod_{i=1}^{m}(x_k - x_i) = x_k^m - e_1 x_k^{m-1} + \dots + (-1)^{m-1} e_{m-1} x_k + (-1)^m e_m = 0$$

and A is generated as an S-module by the products $x_1^{a_1} \dots x_m^{a_m}$, $0 \leq a_k < m$.

Definition 6.1.8 Let M be a module of the graded algebra $A = \sum_{n \geq 0} A^{(n)}$. One says that the module M is *graded* if M is a graded vector space with the same grading as A, (i.e. $M = \sum_{k \geq 0} M^{(k)}$) and

$$A^{(n)} M^{(k)} \subseteq M^{(n+k)}.$$

Hilbert-Serre Theorem 6.1.9 (See [16].) *Let $A = K[x_1, \dots, x_m]$ be the polynomial algebra in m variables graded by assuming that the variable x_i is of degree d_i, $i = 1, \dots, m$. Let $W = \sum_{n \geq 0} W^{(n)}$ be a finitely generated graded A-module. Then the Hilbert series of W*

$$\mathrm{Hilb}(W, t) = \sum_{n \geq 0} \dim W^{(n)} t^n$$

is a rational function of the form

$$\mathrm{Hilb}(W, t) = f(t) \prod_{i=1}^{m} \frac{1}{1 - t^{d_i}},$$

where $f(t) \in \mathbb{Z}[t]$.

Proof. Since W is a finitely generated graded A-module, for each n the homogeneous component of degree n of W is finite dimensional. Hence the Hilbert series of W is well defined. We apply induction on the number m of the generators of $A = K[x_1, \dots, x_m]$. First, let $m = 0$. Then $A = K$ and W is a finite dimensional graded vector space. Therefore, the Hilbert series of W is a polynomial.

Now, let $m > 0$ and let the statement of the theorem be true for $m-1$. The element x_m acts on W by multiplication, which (in this proof only) we denote by μ, i.e. $\mu(w) = x_m w$, $w \in W$. The restriction μ_n of μ on $W^{(n)}$ is a linear transformation of $W^{(n)}$ into $W^{(n+d_m)}$. Let

$$U^{(n)} = \mathrm{Ker}\mu_n,\ \mathrm{Im}\mu_n = \mu(W^{(n)})$$

be respectively the kernel and the image of μ_n. Let

$$U = \sum_{n\geq 0} U^{(n)},\ T = \sum_{n\geq 0} T^{(n+d_m)},\ \text{ where } T^{(n+d_m)} = W^{(n+d_m)}/\mathrm{Im}\mu_n.$$

Both U and T are finitely generated A-modules, because U is a submodule and T is a factor module of W (see Exercise 6.1.6 (iii)). Clearly,

$$x_m U^{(n)} = x_m \mathrm{Ker}\mu_n = 0,$$

$$x_m T^{(n+d_n)} = x_m(W^{(n+d_m)}/\mathrm{Im}\mu_n) = \mathrm{Im}\mu_{n+d_m}/\mathrm{Im}\mu_{n+d_m} = 0.$$

Hence the action of x_m on U and on T is trivial (i.e. U and T are annihilated by x_m) and this means that U and T are graded $K[x_1,\ldots,x_{m-1}]$-modules. Since

$$W^{(n)}/U^{(n)} \cong \mathrm{Im}\mu_n,\ T^{(n+d_m)} = W^{(n+d_m)}/\mathrm{Im}\mu_n,$$

we obtain that

$$\mathrm{dim}W^{(n)} - \mathrm{dim}U^{(n)} = \dim\,\mathrm{Im}\mu_n = \mathrm{dim}W^{(n+d_m)} - \mathrm{dim}T^{(n+d_m)}.$$

Multiplying this equation by t^{n+d_m} and summing with respect to n we get

$$t^{d_m}(\mathrm{Hilb}(W,t) - \mathrm{Hilb}(U,t)) = \mathrm{Hilb}(W,t) - \mathrm{Hilb}(T,t) - g(t),$$

where

$$g(t) = \sum_{n=0}^{d_m-1} \mathrm{dim}W^{(n)}t^n$$

is a polynomial in n with integer coefficients. Hence

$$(1-t^{d_m})\mathrm{Hilb}(W,t) = \mathrm{Hilb}(T,t) - t^{d_m}\mathrm{Hilb}(U,t) + g(t).$$

Since U and T are finitely generated $K[x_1,\ldots,x_{m-1}]$-modules, we apply the inductive assumption and obtain the desired result.

Exercise 6.1.10 If in the notation of Hilbert-Serre-Theorem 6.1.9 all variables x_i are of first degree, show that for n sufficiently large, $\mathrm{dim}W^{(n)}$ is a polynomial in n (with rational coefficients) of degree less or equal to $m-1$. Usually this polynomial is called the *Hilbert polynomial* of W.

Hint. Let $f(t) = \sum_{k=0}^{p} a_k t^k$. Use that $\mathrm{dim}W^{(n)}$ is equal to the coefficient of t^n in $f(t)(1-t)^{-m}$ and

$$\frac{1}{(1-t)^m}=\sum_{n\geq 0}\binom{m+n-1}{m-1}t^n.$$

Hence

$$\dim W^{(n)}=\sum_{k=0}^{p}a_k\binom{m+n-1}{m-1}$$

for $n\geq p$.

Exercise 6.1.11 In the notation of Exercise 6.1.10, let the order of the pole at $t=1$ of the Hilbert series $\mathrm{Hilb}(W,t)=f(t)(1-t)^m$ be equal to d. Show that the degree of the Hilbert polynomial of W is equal to $d-1$.

Exercise 6.1.12 Generalize Hilbert-Serre Theorem 6.1.9 to the case when W is a finitely generated multigraded $K[x_1,\dots,x_m]$-module and prove that there exists a polynomial $f\in\mathbb{Z}[t_1,\dots,t_m]$ such that

$$\mathrm{Hilb}(W,t_1,\dots,t_m)=f(t_1,\dots,t_m)\prod_{i=1}^{m}\frac{1}{1-t_i}.$$

6.2 Nonmatrix Polynomial Identities

In this section we give some applications of commutative algebra to the theory of PI-algebras. The following result is due to Drensky [75] and is based on the idea of Krasilnikov [156] to consider some T-ideals as modules over polynomial algebras. The result is also valid if we consider nonunitary algebras.

Theorem 6.2.1 *Let K be an infinite field and let $W=T(R)$ be the T-ideal of the identities of an algebra R satisfying for some $k\leq 1$ the polynomial identity*

$$[x_1,x_2]\dots[x_{2k-1},x_{2k}]=0.$$

Then for any $m\geq 1$ the Hilbert series

$$\mathrm{Hilb}(F_m(R),t)=\sum_{n\geq 0}\dim F_m(R)^{(n)}t^n$$

of the relatively free algebra

$$F_m(R)=K\langle x_1,\dots,x_m\rangle/(K\langle x_1,\dots,x_m\rangle\cap W)$$

is a rational function of t.

Proof. Let W_k be the T-ideal of $K\langle x_1,\dots,x_m\rangle$ generated by the polynomial identity $[x_1,x_2]\dots[x_{2k-1},x_{2k}]=0$. This means that, as an ideal, W_k is generated by all elements

$$[u_1,u_2]\dots[u_{2k-1},u_{2k}],\ u_i\in K\langle x_1,\dots,x_m\rangle.$$

For simplicity of notation we write W instead of $K\langle x_1,\dots,x_m\rangle\cap W$. We apply induction on k. The base of the induction, $k=1$, follows from Hilbert-Serre Theorem 6.1.9, because $F_m(R)$ is a graded homomorphic image of $K[x_1,\dots,x_m]$. (By Exercise 4.3.7, if we consider unitary algebras, $F_m(R)$ is isomorphic to $K[x_1,\dots,x_m]$.) Now, let $k>1$ and let $W\supset W_k$. Since the following isomorphism of graded vector spaces holds

$$F_m(R)\cong K\langle x_1,\dots,x_m\rangle/W\cong$$

$$\cong K\langle x_1,\dots,x_m\rangle/(W_{k-1}+W)\oplus W_{k-1}/(W_{k-1}\cap W),$$

it is sufficient (why?) to assume that $W_{k-1}\supset W\supset W_k$ and to show that the Hilbert series of W_{k-1}/W is a rational function. We shall consider the case $k=3$. The case $k=2$ is simpler but does not give the idea about the typical difficulties. The general case $k>3$ is similar to $k=3$. In Theorem 5.2.1 we have already seen that W_2/W_3 is spanned as a vector space by the polynomials

$$u_{abc}=x_1^{a_1}\dots x_m^{a_m}[x_{b_1},x_{b_2},\dots,x_{b_p}][x_{c_1},x_{c_2},\dots,x_{c_q}].$$

(We may assume that $b_1>b_2\leq b_3\leq\dots\leq b_p$, $c_1>c_2\leq c_3\leq\dots\leq c_q$, but we do not need this fact.) We consider the polynomial algebra $A=K[\xi_i,\eta_i,\zeta_i\mid i=1,\dots,m]$ in $3m$ variables and define an action of A on W_2/W_3 by

$$\xi_i * x_1^{a_1}\dots x_m^{a_m}[x_{b_1},x_{b_2},\dots,x_{b_p}][x_{c_1},x_{c_2},\dots,x_{c_q}]=$$

$$=x_ix_1^{a_1}\dots x_m^{a_m}[x_{b_1},x_{b_2},\dots,x_{b_p}][x_{c_1},x_{c_2},\dots,x_{c_q}],$$

$$\eta_i * x_1^{a_1}\dots x_m^{a_m}[x_{b_1},x_{b_2},\dots,x_{b_p}][x_{c_1},x_{c_2},\dots,x_{c_q}]=$$

$$=x_1^{a_1}\dots x_m^{a_m}[x_{b_1},x_{b_2},\dots,x_{b_p},x_i][x_{c_1},x_{c_2},\dots,x_{c_q}],$$

$$\zeta_i * x_1^{a_1}\dots x_m^{a_m}[x_{b_1},x_{b_2},\dots,x_{b_p}][x_{c_1},x_{c_2},\dots,x_{c_q}]=$$

$$=x_1^{a_1}\dots x_m^{a_m}[x_{b_1},x_{b_2},\dots,x_{b_p}][x_{c_1},x_{c_2},\dots,x_{c_q},x_i],$$

i.e. ξ_i multiplies from the left the polynomial u_{abc} by x_i; η_i and ζ_i add the variable x_i in the end of the first and the second commutator, respectively. We shall use that the following identities hold in W_2/W_3:

$$y_{\rho(1)}\dots y_{\rho(n)}[z_1,z_2,z_{\sigma(3)},\dots,z_{\sigma(p)}][u_1,u_2,u_{\tau(3)},\dots,u_{\tau(q)}]=$$

$$=y_1\dots y_n[z_1,z_2,z_3,\dots,z_p][u_1,u_2,u_3,\dots,u_q],$$

where $\rho \in S_n$, $\sigma \in S_p$, $\tau \in S_q$ and σ and τ fix 1 and 2. This means that the action of A on W_2/W_3 is well defined and W_2/W_3 is an A-module generated by a finite system of elements

$$[x_{i_1}, x_{i_2}][x_{i_3}, x_{i_4}],\ i_1, i_2, i_3, i_4 = 1, \ldots, m.$$

Since the base field is infinite, W/W_3 is a graded vector subspace of W_2/W_3. Unfortunately, it is not obliged to be an A-submodule. We shall avoid this difficulty in the following way. Let $f = f(x_1, \ldots, x_m) \in W/W_3$,

$$f = \sum \alpha_{abc} x_1^{a_1} \ldots x_m^{a_m} [x_{b_1}, x_{b_2}, \ldots, x_{b_p}][x_{c_1}, x_{c_2}, \ldots, x_{c_q}],\ \alpha_{abc} \in K.$$

Let β be an arbitrary element of K. We fix a variable x_i. Since W is a T-ideal,

$$f(x_1 + \beta[x_1, x_i], \ldots, x_m + \beta[x_m, x_i]) \in W/W_3.$$

Direct calculations show that

$$[x_r + \beta[x_r, x_i], x_s + \beta[x_s, x_i]] = [x_r, x_s] + \beta[x_r, x_s, x_i] + \beta^2[[x_r, x_i], [x_s, x_i]],$$

$$\begin{aligned} f(x_1 + \beta[x_1, x_i], \ldots, x_m + \beta[x_m, x_i]) &= f(x_1, \ldots, x_m) + \\ +\beta \sum \alpha_{abc} x_1^{a_1} \ldots x_m^{a_m} &[x_{b_1}, x_{b_2}, \ldots, x_{b_p}, x_i][x_{c_1}, x_{c_2}, \ldots, x_{c_q}] + \\ +\beta \sum \alpha_{abc} x_1^{a_1} \ldots x_m^{a_m} &[x_{b_1}, x_{b_2}, \ldots, x_{b_p}][x_{c_1}, x_{c_2}, \ldots, x_{c_q}, x_i] + \\ +\beta^2 \sum \alpha_{abc} x_1^{a_1} \ldots x_m^{a_m} &[x_{b_1}, x_{b_2}, \ldots, x_{b_p}, x_i][x_{c_1}, x_{c_2}, \ldots, x_{c_q}, x_i] = \\ &= (1 + \beta(\eta_i + \zeta_i) + \beta^2 \eta_i \zeta_i) * f(x_1, \ldots, x_m). \end{aligned}$$

Applying Vandermonde arguments (K is infinite!) we obtain that

$$(\eta_i + \zeta_i) * f(x_1, \ldots, x_m) \in W/W_3,\ \eta_i \zeta_i * f(x_1, \ldots, x_m) \in W/W_3.$$

Since

$$\xi_i * f(x_1, \ldots, x_m) = x_i f(x_1, \ldots, x_m) \in W/W_3,$$

we establish that W/W_3 is a submodule of W_2/W_3 with respect to the action of the (finitely generated) subalgebra S of A generated by

$$\xi_i,\ \eta_i + \zeta_i,\ \eta_i \zeta_i,\ i = 1, \ldots, m.$$

Since $\eta_i + \zeta_i$ and $\eta_i \zeta_i$ generate the algebra of symmetric polynomials in two variables η_i, ζ_i, by Exercise 6.1.7 (and its hint!) we obtain that A is a finitely generated S-module. Now W_2/W_3 is a finitely generated A-module, hence W_2/W_3 is finitely generated also as an S-module (why?). Therefore, its factor module W_2/W is also finitely generated. The conclusion of the theorem follows from Hilbert-Serre Theorem 6.1.9, since the usual grading of the free algebra is the same as the grading of S defined by

$$\deg \xi_i = \deg(\eta_i + \zeta_i) = 1,\ \deg(\eta_i \zeta_i) = 2.$$

The proof of the following corollary of Theorem 6.2.1 follows immediately from the theorem of Latyshev (see Remark 5.2.2) that any finitely generated PI-algebra which satisfies a nonmatrix polynomial identity, satisfies also for some $k \geq 1$ the polynomial identity

$$[x_1, x_2] \dots [x_{2k-1}, x_{2k}] = 0.$$

Although we do not know whether R itself is finitely generated, the relatively free algebra $F_m(R)$ is m-generated and we may apply the theorem.

Corollary 6.2.2 *If the base field K is infinite and the algebra R satisfies a nonmatrix polynomial identity, then the Hilbert series* $\mathrm{Hilb}(F_m(R), t)$ *is a rational function.*

Exercise 6.2.3 Generalize Theorem 6.2.1 and Exercise 6.2.2 and prove the rationality of the Hilbert series $\mathrm{Hilb}(F_m(R), t_1, \dots, t_m)$, where the base field is infinite and the algebra R satisfies a nonmatrix polynomial identity.

Remark 6.2.4 The action of the polynomial algebra in the proof of Theorem 6.2.1 was first used by Krasilnikov [156] to show that over a field of characteristic 0, every Lie algebra which satisfies the polynomial identity

$$[[x_1, x_2], \dots, [x_{2k-1}, x_{2k}]] = 0$$

has a finite basis for its polynomial identities. His proof is based on the following approach. Let W_k be the T-ideal of the free Lie algebra L_m consisting of the polynomial identities in m variables of the Lie algebra of $k \times k$ upper triangular matrices. Then for the T-ideals $W \subset L_m$ such that $W_{k-1} \supset W \supset W_k$, the vector space W/W_k is a submodule of a finitely generated module of a finitely generated commutative algebra (as in Theorem 6.2.1). Besides, Krasilnikov used that over a field of characteristic 0, modulo the T-ideal $T(R)$ of the polynomial identities of a finite dimensional algebra R, every polynomial identity is equivalent to a system of polynomial identities in not more than $\dim R$ variables. Finally, since W_k is the T-ideal of the Lie algebra of $k \times k$ upper triangular matrices, it is sufficient to consider $m = \frac{1}{2}n(n+1)$, the dimension of this algebra, and to apply the fact that the T-ideals W such that $W_{k-1} \supset W \supset W_k$ satisfy the ascending chain condition (and the ideal W_k itself is finitely generated as a T-ideal). The same approach works for associative algebras ($\mathrm{char} K = 0$) satisfying the polynomial identity

$$[x_1, x_2] \dots [x_{2k-1}, x_{2k}] = 0.$$

The associative case was initially handled by Latyshev [164] and Genov [110, 112] in 1976 using another technique. As we have already mentioned in Chapter 3 (see Theorem 3.1.4), in 1987 Kemer proved that every associative algebra over a field of characteristic 0 has a finite basis for its polynomial identities.

Recently, Theorem 6.2.1 has been generalized by Belov for all relatively free associative algebras.

Theorem 6.2.5 (Belov [26]) *For any relatively free associative algebra* $F_m(R)$ *over an infinite field* K, *the Hilbert series* $\mathrm{Hilb}(F_m(R),t)$ *is a rational function of* t.

The proof of Belov uses ideas of commutative algebras similar to these in the proof of Theorem 6.2.1 and is essentially based on the results of Kemer on structure theory of T-ideals. (See Chapter 8 for a short description of the results of Kemer.) In particular, Belov uses the positive solution of the Specht problem in characteristic 0 and some weaker version of this result in positive characteristic.

Exercise 6.2.6 Show that if a T-ideal W in the free Lie algebra L over an infinite field K contains the polynomial identity

$$[[x_1,x_2],\ldots,[x_{2k-1},x_{2k}]]=0$$

then the Hilbert series of $L_m/(L_m\cap W)$ is rational.

Hint. We give hints for $k=3$ only in order to use the notation of the proof of Theorem 6.2.1 (e.g. W_3 is an ideal of the free associative algebra of rank m). Assume that $L_m\subset K\langle x_1,\ldots,x_m\rangle$. Consider the case $W\supset W_2$. Let

$$S_0=K[\eta_i+\zeta_i,\eta_i\zeta_i\mid i=1,\ldots,m].$$

Show that the set of the proper polynomials in W_2/W_3 is a finitely generated S_0-module and $(L_m\cap W_2)/(L_m\cap W_3)$ and $(W+W_3)/W_3$ are its submodules. Hence $(L_m\cap W_2)/W$ is a finitely generated graded S_0-module.

6.3 Commutative and Noncommutative Invariant Theory

In this section we give some idea of classical (commutative) invariant theory and its noncommutative generalization. If the reader is interested in this topic, we forward him or her to the books by Dieudonné and Carrell [60], Springer [246] and Sturmfels [247] (for classical invariant theory), to the book by Formanek [108] for the invariant theory of matrices, and to the book by Kharchenko [147] and the survey articles by Formanek [106] and Drensky [84] for the noncommutative generalizations. Till the end of the section we assume that the base field K is algebraically closed (although most of the considerations do not require this) and of characteristic 0.

Exercise 6.3.1 Let C be a commutative algebra generated by a finite set $\{c_1, \ldots, c_m\}$ and let S be a subalgebra of C such that the generators of C are *integral* over S. This means that for each c_i there exists an $n_i \in \mathbb{N}$ and $a_{ij} \in S$, such that

$$c_i^{n_i} + a_{i1}c_i^{n_i-1} + \ldots + a_{i,n_i-1}c_i + a_{in_i} = 0.$$

Show that the algebra S is finitely generated.

Hint. Let S_0 be the subalgebra of S generated by all coefficients a_{ij} appearing in the above equations. Show that every power of c_i is in the S_0-submodule of C generated by $1, c_i, \ldots, c_i^{n_i-1}$. Hence, as an S_0-module, C is generated by the products $c_1^{p_1} \ldots c_m^{p_m}$, $0 \leq p_i \leq n_i - 1$. Since S_0 is noetherian (as a homomorphic image of $K[x_{ij} \mid j = 1, \ldots, n_i, i = 1, \ldots, m]$), every S_0-submodule of C is finitely generated. Let S be generated as an S_0-module by $b_1, \ldots, b_n$. Then, as an algebra, S is generated by all a_{ij}, b_k.

Exercise 6.3.2 Let C be a finitely generated commutative algebra, which is a finitely generated module of a subalgebra S. Show that S is also finitely generated as an algebra.

Hint. If C is generated as an algebra by $c_1, \ldots, c_m$, show that there exist positive integers $n_1, \ldots, n_m$ such that C is generated as an S-module by $c_1^{p_1} \ldots c_m^{p_m}$, $0 \leq p_i < n_i$. Then modify the proof of Exercise 6.3.1.

Definition 6.3.3 Let C be a vector space or an algebra and let G be a group which acts on C as a group of automorphisms. (This means that we have a homomorphism $G \longrightarrow \mathrm{Aut}C$.) We denote by C^G the set of the *fixed elements* in C, i.e.

$$C^G = \{c \in C \mid g(c) = c \text{ for all } g \in G\}.$$

If G is a subgroup of the general linear group $GL_m(K)$ with its canonical action on the vector space V_m with basis $\{x_1, \ldots, x_m\}$, we extend the action of G on the polynomial algebra $K[x_1, \ldots, x_m]$ by

$$g(f(x_1, \ldots, x_m)) = f(g(x_1), \ldots, g(x_m)),\ f \in K[x_1, \ldots, x_m], g \in G.$$

The *algebra of invariants* of G is the algebra $K[x_1, \ldots, x_m]^G$. Similarly we define the action of $GL_m(K)$ on the free algebra $K\langle x_1, \ldots, x_m\rangle$ and on relatively free algebras $F_m(R)$, where R is a PI-algebra, and use the same notation $K\langle x_1, \ldots, x_m\rangle^G$ and $F_m(R)^G$ for the invariants of a subgroup G of $GL_m(K)$. (The action of $GL_m(K)$ on $F_m(R) = K\langle x_1, \ldots, x_m\rangle/T(R)$ is well defined because $T(R)$ is a T-ideal.)

Remark 6.3.4 In classical invariant theory usually one considers the action of $GL_m(K)$ on K^m and treats the polynomials in $K[x_1, \ldots, x_m]$ as functions, i.e. the action of $GL_m(K)$ on $K[x_1, \ldots, x_m]$ is given by

$$(gf)(\xi) = f(g^{-1}(\xi)), f \in K[x_1, \ldots, x_m], \xi = (\xi_1, \ldots, \xi_m) \in K^m.$$

This means that $V_m = \mathrm{span}\{x_1, \ldots, x_m\}$ is the dual module of the $GL_m(K)$-module K^m. We prefer our Definition 6.3.3 because it works better for noncommutative algebras. In order to fit it to the classical definition we have to replace K^m with its dual $GL_m(K)$-module $(K^m)^*$.

We start with the problem of finite generation of the algebra of invariants.

Exercise 6.3.5 Let G be a finite group of automorphisms of a commutative algebra C. Show that every element of C is integral over the algebra of fixed points C^G.

Hint. Let $|G| = n$. Show that for every $c \in C$

$$\prod_{g \in G} (c - g(c)) = c^n + a_1 c^{n-1} + \ldots + a_{n-1} c + a_n = 0,$$

where $a_1, \ldots, a_n \in C^G$.

Exercise 6.3.6 Prove the theorem of Emmy Noether [194] that the algebra of invariants of a finite group G acting on the polynomial algebra in m variables is finitely generated.

Hint. Apply Exercises 6.3.1 and 6.3.5.

Remark 6.3.7 One of the most famous problems in classical invariant theory is the 14-th problem of Hilbert whether the algebra of invariants $K[x_1, \ldots, x_m]^G$ is finitely generated for any *linear group* (= subgroup of $GL_m(K)$). The negative answer was given by Nagata [189]. An exposition of his result can be found in the book by Dieudonné and Carrell [60]. An approach from the point of view of derivations of polynomial algebras is given in the book by Nowicki [196]. In the paper [211] (see also his book [212]), Popov described the groups allowing nonfinitely generated algebras of invariants.

Exercise 6.3.8 Let $U_k(K)$ be the algebra of $k \times k$ upper triangular matrices. Show that for $k = 2$ and for any finite linear group $G \subset GL_m(K)$, the algebra of invariants $F_m(U_2(K))^G$ in the relatively free algebra $F_m(U_2(K))$ is finitely generated.

Hint. Let $\bar{f}_1, \ldots, \bar{f}_n$ generate the algebra $K[x_1, \ldots, x_m]^G$. Consider the canonical homomorphism $F_m(U_2(K)) \longrightarrow K[x_1, \ldots, x_m]$. If $f_1, \ldots, f_n$ are preimages of $\bar{f}_1, \ldots, \bar{f}_n$, then

$$h_i = \frac{1}{|G|} \sum_{g \in G} g(f_i) \in F_m(U_2(K))^G,$$

i.e. we *lift* the invariants of the polynomial algebra to invariants of $F_m(U_2(K))$. The invariants can be lifted also using Maschke Theorem 12.1.6 that every finite dimensional G-module is a direct sum of its irreducible submodules (how?). Write the elements of the *commutator ideal* $F' = F[F, F]F$ of $F = F_m(U_2(K))$ in the form

$$\sum \alpha_{abc} x_1^{a_1} \dots x_m^{a_m} [x_{b_1}, x_{b_2}] x_1^{c_1} \dots x_m^{c_m},\ \alpha_{abc} \in K.$$

Show that F' is a finitely generated $S = K[y_1, \dots, y_n, z_1, \dots, z_m]$-module with action defined by

$$(y_1^{p_1} \dots y_n^{p^n} z_1^{q_1} \dots z_n^{q_n}) * (x_1^{a_1} \dots x_m^{a_m} [x_{b_1}, x_{b_2}] x_1^{c_1} \dots x_m^{c_m}) =$$
$$= h_1^{p_1} \dots h_n^{p^n} x_1^{a_1} \dots x_m^{a_m} [x_{b_1}, x_{b_2}] x_1^{c_1} \dots x_m^{c_m} h_1^{q_1} \dots h_n^{q^n}.$$

Show that any set of generators of the S-module $(F')^G$ together with $h_1, \dots, h_n$ generates the whole algebra $F_m(U_2(K))^G$.

Exercise 6.3.9 Explain why the hints to Exercise 6.3.8 do not work for $F_m(U_k(K))^G$, $k > 2$.

Exercise 6.3.10 Let $F = F_2(U_3(K))$ and let $G = \langle g \rangle$, where

$$g(x_1) = x_1,\ g(x_2) = -x_2.$$

Show that the algebra of invariants F^G is not finitely generated.

Hint. Show that F^G is generated by x_1 and $x_2 x_1^n x_2$, $n = 0, 1, 2, \dots$ Let R be the subalgebra of F^G generated by x_1 and $x_2 x_1^n x_2$, $n = 0, 1, \dots, p$, and let $R = F^G$. As a relatively free algebra, F is multihomogeneous. Hence

$$x_2 x_1^{p+1} x_2 = \sum_{j=0}^{p} \left(\sum_{i+k=p-j+1} \alpha_{ijk} x_1^i x_2 x_1^j x_2 x_1^k \right),\ \alpha_{ijk} \in K.$$

Therefore, this is a polynomial identity for the algebra of 3×3 upper triangular matrices. Show that this is not true. For example, verify this identity on $\bar{x}_1 = e_{22}$, $\bar{x}_2 = e_{12} + e_{23}$.

Exercise 6.3.11 Show that if $F_m(R)$, $m > 1$, is a relatively free algebra such that $F_m(R)^G$ is finitely generated for every finite linear group G, then R satisfies a polynomial identity of the form

$$x_2 x_1^p x_2 + \sum_{j=0}^{p-1} \left(\sum_{i+k=p-j} \alpha_{ijk} x_1^i x_2 x_1^j x_2 x_1^k \right) = 0,\ \alpha_{ijk} \in K.$$

Hint. Choose $g \in GL_m(K)$,

$$g(x_2) = -x_2,\ g(x_i) = x_i,\ i \neq 2.$$

Follow the instructions to Exercise 6.3.10.

Exercise 6.3.12 Let R be the algebra of 3×3 upper triangular matrices of the form

$$\begin{pmatrix} \alpha & * & * \\ 0 & \beta & * \\ 0 & 0 & \alpha \end{pmatrix},\ \alpha, \beta \in K.$$

Find a finite linear group G, such that $F_m(R)^G$ is not finitely generated.

Hint. Use Exercise 6.3.11.

Exercise 6.3.13 Let $C = K[t]/(t^3)$, i.e. $t^3 = 0$ in C, and let R be the subalgebra of the algebra $M_2(C)$ of 2×2 matrices with entries from C, consisting of all matrices of the form

$$\begin{pmatrix} a_{11}(t) & ta_{12}(t) \\ ta_{21}(t) & a_{22}(t) \end{pmatrix},\ a_{ij}(t) \in C.$$

Show that $F_m(R)^G$, $m > 1$, is not finitely generated for some finite groups G.

Remark 6.3.14 Kharchenko [146] proved that $F_m(R)^G$, $m > 1$, is finitely generated for every finite linear group G if and only if $F_m(R)$ is *weakly noetherian*, i.e. noetherian with respect to two-sided ideals. Lvov [173] obtained that this happens if and only if R satisfies a polynomial identity of the form in Exercise 6.3.11. There are many other equivalent conditions given e.g. in the survey articles by Formanek [106], Drensky [84] and by Kharlampovich and Sapir [148]. (Do not be afraid of the title of the latter article! It contains not only algorithmic problems of algebra.) Most of the above exercises are restatements or partial cases of these conditions.

Remark 6.3.15 There is a large class of linear groups including the *reductive* and the *classical groups* which are infinite but nevertheless their algebras of (commutative) invariants are finitely generated. These groups satisfy the so called *Hilbert-Nagata condition.* See the book by Dieudonné and Carrell [60] for the commutative and the paper by Domokos and Drensky [68] for the noncommutative case.

Remark 6.3.16 Considering the invariants of a finite linear group acting on the free associative algebra, the only case when the algebra $K\langle x_1, \ldots, x_m\rangle^G$ is finitely generated is when G is cyclic and acts by scalar multiplication, i.e. $G = \langle g\rangle$ and $g(x_i) = \xi x_i$, where $\xi^n = 1$. This is a result of Dicks and Formanek [58] and Kharchenko [144]. On the other hand, Kharchenko [144] proved that for G finite, there is a Galois correspondence between the subgroups of G and the free subalgebras F of $K\langle x_1,\ldots,x_m\rangle$ containing $K\langle x_1,\ldots,x_m\rangle^G$. This correspondence is given by

$$H \longrightarrow K\langle x_1,\ldots,x_m\rangle^H,\ H < G.$$

Remark 6.3.17 If G is a finite linear group, then it is easy to see (prove it!) that the *transcendence degree* of $K[x_1,\ldots,x_m]^G$ (i.e. the maximal number of algebraically independent over K elements) is equal to m. This means that $K[x_1,\ldots,x_m]^G$ contains a subalgebra isomorphic to the polynomial algebra in m variables. The famous theorem of Shephard-Todd [236] and Chevalley [49] gives that

$$K[x_1,\ldots,x_m]^G \cong K[x_1,\ldots,x_m]$$

if and only if G is generated by *pseudo-reflections* (i.e. diagonalizable linear transformations of V_m which have 1 as an $m-1$-multiple eigenvalue). Typical examples of such groups are the symmetric groups which are generated by reflections (in the usual geometric sense). The corresponding noncommutative analogue of this result was proved by Domokos [67]. He showed that if G is a nontrivial finite linear group, then $F_m(R)^G \cong F_m(R)$, $m > 1$, if and only if R satisfies the polynomial identity $[x_1,x_2,x_3] = 0$ and G is generated by pseudo-reflections. This shows that the polynomial identities of the Grassmann algebra are very close to the commutative law.

Remark 6.3.18 Concerning free Lie algebras, Bryant [40] showed that L_m^G is never finitely generated ($|G| > 1$, $m > 1$). We have mentioned that one of the main differences between associative and Lie PI-algebras is that the associative PI-algebras have good properties which make them close to commutative and finite dimensional algebras and this is not always true for Lie algebras. Such an example can be also found in noncommutative invariant theory. If $|G| > 1$ and the Lie algebra R is not nilpotent, then $F_m(R)^G$, $m > 1$, is not finitely generated (see the paper by Drensky [83]).

In the next several exercises we discuss the Hilbert series of the invariants of finite linear groups. We assume that G is a finite group acting as a group of invertible linear operators on a finite dimensional vector space W. Since every element g of G is of finite order and $\mathrm{char}K = 0$, the matrix of g is diagonalizable (prove it!). We denote the eigenvalues of g by $\xi_i(g)$, $i = 1,2,\ldots,\dim W$.

Exercise 6.3.19 Let the linear operator ϕ of W be defined by

$$\phi = \frac{1}{|G|}\sum_{h\in G} h.$$

Show that:

(i) $\phi^2 = \phi$.

(ii) An element $w \in W$ is G-invariant (i.e. $g(w) = w$ for all $g \in G$) if and only if $w \in \mathrm{Im}\phi = \phi(W)$.

(iii) $\dim W^G = \mathrm{tr}_W(\phi)$, the trace of the operator ϕ.

Hint. (i)-(ii) Use that

$$g\sum_{h\in G} h = \sum_{h\in H} h = \left(\sum_{h\in G} h\right) g,\ g \in G.$$

(iii) Since $\phi^2 = \phi$, use that $W = \mathrm{Ker}\phi \oplus \mathrm{Im}\phi$ and ϕ acts identically on $\mathrm{Im}\phi$.

Exercise 6.3.20 Let $g \in GL_m(K)$ be a diagonalizable linear operator acting on $V_m = \mathrm{span}\{x_1, \ldots, x_m\}$. Let $\xi_1, \ldots, \xi_m$ be the eigenvalues of g.

(i) Show that the eigenvalues of g acting on the vector space of all homogeneous polynomials of degree n are equal to

$$\xi_1^{a_1} \ldots \xi_m^{a_m},\ a_1 + \ldots + a_m = n.$$

(ii) Let W be a multihomogeneous finite dimensional vector subspace of the free algebra $K\langle x_1, \ldots, x_m\rangle$ and let W be invariant under the action of the general linear group. Show that the Hilbert series of W

$$\mathrm{Hilb}(W, t_1, \ldots, t_m) = \sum \dim W^{(n_1,\ldots,n_m)} t_1^{n_1} \ldots t_m^{n_m}$$

is a symmetric polynomial in $t_1, \ldots, t_m$. Prove that the trace of g acting on W is

$$\mathrm{tr}_W(g) = \mathrm{Hilb}(W, \xi_1, \ldots, \xi_m).$$

Hint. (i) The statement is obvious if $x_1, \ldots, x_m$ are the eigenvectors of g. We may change linearly the variables (and g still will have the same eigenvalues) in such a way that to be in the desired situation.

(ii) Apply to W the linear operator $\bar\sigma \in GL_m(K)$ defined by $\bar\sigma(x_i) = x_{\sigma(i)}$, $\sigma \in S_m$, and, using that $\sigma(W) = W$, prove that

$$\mathrm{Hilb}(W, t_1, \ldots, t_m) = \mathrm{Hilb}(W, t_{\sigma(1)}, \ldots, t_{\sigma(m)}).$$

For the trace $\mathrm{tr}_W(g)$ apply the arguments of (i).

Exercise 6.3.21 If $G \subset GL_m(K)$, show that for any relatively free algebra $F = F_m(R)$ (including $F = K[x_1, \ldots, x_m]$ and $F = K\langle x_1, \ldots, x_m\rangle$), the algebra of invariants F^G is graded (but not always multigraded).

Hint. If $f = \sum_{i=0}^k f_k$, $f_i \in F^{(i)}$, then $f \in F^G$ if and only if $f_i \in F^G$, $i = 0, 1, \ldots, k$.

Exercise 6.3.21 shows that the Hilbert series of the algebra of the invariants $F = F_m(R)^G$ is well defined and

$$\mathrm{Hilb}(F^G, t) = \sum_{n\geq 0} \dim(F^{(n)})^G t^n.$$

Exercise 6.3.22 Prove the following theorem of Formanek [106]. If G is a finite subgroup of $GL_m(K)$, the eigenvalues of $g \in G$ are $\xi_i(g)$, $i = 1, \ldots, m$, and $F = F_m(R)$ is a relatively free algebra, then

$$\mathrm{Hilb}(F^G, t) = \frac{1}{|G|} \sum_{g\in G} \mathrm{Hilb}(F, \xi_1(g)t, \ldots, \xi_m(g)t).$$

Hint. Apply Exercise 6.3.20.

In the next exercises we assume that $G \subset GL_m(K)$ is a finite group. We denote by $\det(g)$ and $\mathrm{tr}(g)$ the determinant and the trace of g as an $m \times m$ matrix. The hint to all exercises is the same: Find the corresponding result which gives the Hilbert series of the relatively free algebra under consideration and apply the theorem of Formanek in Exercise 6.3.22.

Exercise 6.3.23 Prove the Molien formula [185]

$$\mathrm{Hilb}(K[x_1, \ldots, x_m]^G, t) = \frac{1}{|G|} \sum_{g\in G} \frac{1}{\det(1 - tg)}.$$

Exercise 6.3.24 Prove the theorem of Dicks and Formanek [58]

$$\mathrm{Hilb}(K\langle x_1, \ldots, x_m\rangle^G, t) = \frac{1}{|G|} \sum_{g\in G} \frac{1}{1 - t\,\mathrm{tr}(g)}.$$

Exercise 6.3.25 Prove that for the Grassmann algebra E

$$\mathrm{Hilb}(F_m(E)^G, t) = \frac{1}{|G|} \sum_{g\in G} \frac{\det(1 + tg)}{\det(1 - tg)}.$$

Exercise 6.3.26 Show that if R satisfies a nonmatrix polynomial identity, then the Hilbert series $\mathrm{Hilb}(F_m(R)^G, t)$ is a rational function. (This means that, although the algebra $F_m(R)^G$ may be not finitely generated, its Hilbert series is still nice. The same result holds for relatively free algebras satisfying a nonmatrix polynomial identity and (not necessarily finite) groups G satisfying the Hilbert-Nagata condition, see the paper by Domokos and Drensky [68].)

Exercise 6.3.27 Generalize the theorem of Formanek in Exercise 6.3.22 to the Hilbert series of the vector spaces of the fixed points of finite linear groups acting on $GL_m(K)$-submodules and factor modules of $K\langle x_1,\ldots,x_m\rangle$.

Hint. Do not forget to prove that the submodules and factor modules of $K\langle x_1,\ldots,x_m\rangle$ are multigraded vector spaces (Vandermonde arguments!).

7. Polynomial Identities of the Matrix Algebras

Since the matrix algebras are considered of great importance both for mathematics and its applications, from the very origins of the theory of PI-algebras, the polynomial identities of matrices have been an attractive object for study. We start this chapter with two proofs of the famous Amitsur-Levitzki theorem which states that the $k \times k$ matrix algebra satisfies the standard identity of degree $2k$. We also discuss other polynomial identities for matrices. Then we introduce the generic matrix algebras which are important not only for PI-theory itself but also for numerous applications to other branches of mathematics such as invariant theory and theory of division algebras. Many important results in PI-theory were established or their proofs were significantly simplified using central polynomials for the matrix algebras. Here we give two essentially different approaches to central polynomials due respectively to Formanek and Razmyslov. For further reading on all these topics we refer to the books by Procesi [213], Jacobson [128], Rowen [231] and Formanek [108].

7.1 The Amitsur-Levitzki Theorem

Recall that the standard polynomial of degree n is

$$s_n(x_1, \ldots, x_n) = \sum_{\sigma \in S_n} (\mathrm{sign}\sigma) x_{\sigma(1)} \ldots x_{\sigma(n)}.$$

It is multilinear and *alternating*, i.e.

$$s_n(\ldots, x_i, \ldots, x_j, \ldots) = -s_n(\ldots, x_j, \ldots, x_i, \ldots),\ s_n(\ldots, x_i, \ldots, x_i, \ldots) = 0.$$

The Capelli polynomial in n skew-symmetric variables is

$$d_n(x_1, \ldots, x_n; y_1, \ldots, y_{n+1}) = \sum_{\sigma \in S_n} (\mathrm{sign}\sigma) y_1 x_{\sigma(1)} y_2 \ldots y_n x_{\sigma(n)} y_{n+1}.$$

Exercise 7.1.1 Show that the standard identity of degree $n+1$ is a consequence of the standard identity of degree n.

Hint. Show that

$$s_{n+1}(x_1,\ldots,x_{n+1})=\sum_{i=1}^{n+1}(-1)^{i-1}x_is_n(x_1,\ldots,\hat{x}_i,\ldots,x_{n+1}),$$

where $\hat{x}_i$ means that x_i does not participate in the expression.

Exercise 7.1.2 (i) Show that the $k\times k$ matrix algebra $M_k(K)$ does not satisfy a polynomial identity of degree less than $2k$.

(ii) Show that $M_k(K)$ does not satisfy the Capelli identity $d_{k^2}=0$ (but, as we have seen in Examples 2.1.3 (ii), it satisfies $d_{k^2+1}=0$).

(iii) Show that if $f(x_1,\ldots,x_{2k})=0$ is a multilinear polynomial identity of degree $2k$ for the algebra $M_k(K)$, then

$$f(x_1,\ldots,x_{2k})=\alpha s_{2k}(x_1,\ldots,x_{2k}),\ \alpha\in K.$$

Hint. (i) If $M_k(K)$ satisfies a polynomial identity of degree $n<2k$, then it satisfies also a multilinear identity

$$f(x_1,\ldots,x_n)=\sum_{\sigma\in S_n}\alpha_\sigma x_{\sigma(1)}\ldots x_{\sigma(n)}=0,\ \alpha_\sigma\in K,$$

of degree n. Use *staircase* arguments:

$$f(e_{11},e_{12},e_{22},e_{23},\ldots,e_{pq})=\alpha_\varepsilon e_{1p},$$

where $p=q$ or $p=q-1$ depending on the parity of n and ε is the identity permutation. Hence $\alpha_\varepsilon=0$. Modify these arguments for α_σ, where σ is an arbitrary permutation in S_n.

(ii) Order the matrix units e_{ij} in an arbitrary way

$$e_{p_1q_1},e_{p_2q_2},\ldots,e_{p_{k^2}q_{k^2}}.$$

Show that

$$d_{k^2}(e_{p_1q_1},e_{p_2q_2},\ldots,e_{p_{k^2}q_{k^2}};e_{1p_1},e_{q_1p_2},\ldots,e_{q_{k^2-1}p_{k^2}},e_{q_{k^2}1})=e_{11}.$$

(iii) Let

$$f(x_1,\ldots,x_{2k})=\sum_{\sigma\in S_{2k}}\alpha_\sigma x_{\sigma(1)}\ldots x_{\sigma(2k)}=0.$$

As in (i), replace $x_1,\ldots,x_{2k}$ by matrix units. For example,

$$f(e_{11},e_{11},e_{12},e_{22},\ldots,e_{k-1,k},e_{kk})=(\alpha_\varepsilon+\alpha_{(12)})e_{1k}=0$$

and, similarly, $\alpha_\sigma+\alpha_\tau=0$ if τ is obtained from σ by transposition of two consequent elements $\tau(p)$ and $\tau(p+1)$. Derive from here that $\alpha_\sigma=(\text{sign}\sigma)\alpha$.

There is a nice graph theoretic interpretation of the multiplication of matrix units which is very useful for simplifying of calculations. For any set of matrix units

$$P = \{e_{ij} \mid (i,j) \in I\} \subset M_k(K)$$

we define an oriented graph $G = G(P)$ with set of vertices $V(G) = \{1, 2, \ldots, k\}$ and set of oriented edges $\{(i,j) \mid (i,j) \in I\}$. Then a product of matrix units $e_{i_1 j_1} \ldots e_{i_q j_q}$ is not zero if and only if the edges $(i_1, j_1), \ldots, (i_k, j_k)$ form a path in the graph G.

For example, if $k = 4$, then Fig. 7.1 gives the oriented graph corresponding to the set of vertices (with their multiplicities) $\{e_{11}, e_{11}, e_{13}, e_{14}, e_{21}, e_{32}\}$.

Fig. 7.1. The graph corresponding to $\{e_{11}, e_{11}, e_{13}, e_{14}, e_{21}, e_{32}\}$

If we denote by v_1 and v_2 the two different copies of e_{11}, then all paths which go exactly once through all edges are

$$1 \xrightarrow{v_1} 1 \xrightarrow{v_2} 1 \to 3 \to 2 \to 1 \to 4, \qquad 1 \xrightarrow{v_2} 1 \xrightarrow{v_1} 1 \to 3 \to 2 \to 1 \to 4,$$

$$1 \xrightarrow{v_1} 1 \to 3 \to 2 \to 1 \xrightarrow{v_2} 1 \to 4, \qquad 1 \xrightarrow{v_2} 1 \to 3 \to 2 \to 1 \xrightarrow{v_1} 1 \to 4,$$

$$1 \to 3 \to 2 \to 1 \xrightarrow{v_1} 1 \xrightarrow{v_2} 1 \to 4, \qquad 1 \to 3 \to 2 \to 1 \xrightarrow{v_2} 1 \xrightarrow{v_1} 1 \to 4.$$

Since the dimension of the matrix algebra $M_k(K)$ is equal to k^2, it satisfies the standard identity $s_{k^2+1} = 0$. The following theorem gives the optimal degree of a standard identity for $M_k(K)$.

Amitsur-Levitzki Theorem 7.1.3 *The $k \times k$ matrix algebra $M_k(K)$ satisfies the standard identity of degree $2k$*

$$s_{2k}(x_1, \ldots, x_{2k}) = 0.$$

Together with Exercise 7.1.2, the Amitsur-Levitzki theorem gives that, up to a multiplicative constant, the standard identity is the only multilinear polynomial identity of degree $2k$ for $M_k(K)$. There are five essentially different proofs of the Amitsur-Levitzki theorem. The original proof [11] is based on inductive combinatorial arguments; with some technical improvements it can be found for example in the book by Passmann [205, p. 175]. There is a graph theoretic proof of Swan [249] which treats the nonzero products of matrix units as paths of *Eulerian graphs* (oriented graphs such that there exists a path passing through all edges exactly once). Recently, the basic idea of this proof was used by Szigeti, Tuza and Révész [250] to give another proof and to show that the matrix algebras satisfy also other interesting polynomial identities. Kostant [152] gave a cohomological proof. Here we give two other proofs of the Amitsur-Levitzki theorem due to Razmyslov [219] and Rosset [230]. We start with the proof of Razmyslov.

Exercise 7.1.4 Show that the validity of the Amitsur-Levitzki theorem for $M_k(\mathbb{Q})$ implies its validity for $M_k(K)$ over any field K.

Hint. If $r_p = \sum_{i,j=1}^{k} \alpha_{ij}^{(p)} e_{ij}$, $\alpha_{ij}^{(p)} \in K$, $p = 1, \dots, 2k$, are matrices in $M_k(K)$, then $s_{2k}(r_1, \dots, r_{2k})$ is a linear combination of $s_{2k}(e_{i_1 j_1}, \dots, e_{i_{2k} j_{2k}})$ and is equal to 0 because we have assumed that the theorem holds for $M_k(\mathbb{Z}) \subset M_k(\mathbb{Q})$.

Exercise 7.1.5 Let the eigenvalues of the matrix $a \in M_k(K)$ be equal to $\xi_i, \dots, \xi_k$ and let $e_q(\xi_1, \dots, \xi_k)$ be the q-th elementary symmetric polynomial in $\xi_1, \dots, \xi_k$. Then

$$a^k + \sum_{q=1}^{k} (-1)^q e_q(\xi_1, \dots, \xi_k) a^{k-q} = 0,$$

$$\operatorname{tr}(a^q) = \xi_1^q + \dots + \xi_k^q.$$

Hint. Apply the Cayley-Hamilton theorem to the matrix a written in its Jordan normal form.

The Razmyslov Proof of Amitsur-Levitzki Theorem 7.1.6 [219] We give the proof for 2×2 matrices. The general case is similar with additional technical difficulties only. By Exercise 7.1.4 we assume that $K = \mathbb{Q}$. Let

$$p_q(\xi_1, \dots, \xi_k) = \xi_1^q + \dots + \xi_k^q.$$

The Newton formulas give that for $q \leq k$

$$p_q - e_1 p_{q-1} + e_2 p_{q-2} + \dots + (-1)^{q-1} e_{q-1} p_1 + (-1)^q q e_q = 0.$$

We can express $e_q(\xi_1, \dots, \xi_k)$ as polynomials of $p_q(\xi_1, \dots, \xi_k)$. In our case

$$e_2(\xi_1, \xi_2) = \xi_1 \xi_2 = \frac{1}{2}((\xi_1 + \xi_2)^2 - (\xi_1^2 + \xi_2^2)) = \frac{1}{2}(p_1^2(\xi_1, \xi_2) - p_2(\xi_1, \xi_2)).$$

For any 2×2 matrix a with eigenvalues ξ_1, ξ_2, the Cayley-Hamilton theorem gives that

$$a^2 - e_1(\xi_1,\xi_2)a + e_2(\xi_1,\xi_2)e = 0,$$

$$a^2 - p_1(\xi_1,\xi_2)a + \frac{1}{2}(p_1^2(\xi_1,\xi_2) - p_2(\xi_1,\xi_2))e = 0,$$

$$a^2 - \mathrm{tr}(a)a + \frac{1}{2}(\mathrm{tr}^2(a) - \mathrm{tr}(a^2))e = 0.$$

Hence

$$x^2 - \mathrm{tr}(x)x + \frac{1}{2}(\mathrm{tr}^2(x) - \mathrm{tr}(x^2))e = 0$$

is a *trace polynomial identity* for $M_2(K)$. We linearize this identity (prove that we can linearize not only ordinary but also trace polynomial identities!) and obtain the identity

$$(y_1y_2 + y_2y_1) - (\mathrm{tr}(y_1)y_2 + \mathrm{tr}(y_2)y_1)+$$

$$+\frac{1}{2}((\mathrm{tr}(y_1)\mathrm{tr}(y_2) + \mathrm{tr}(y_2)\mathrm{tr}(y_1)) - \mathrm{tr}(y_1y_2 + y_2y_1))e = 0,$$

and, since

$$\mathrm{tr}(y_1)\mathrm{tr}(y_2) = \mathrm{tr}(y_2)\mathrm{tr}(y_1),\ \mathrm{tr}(y_1y_2) = \mathrm{tr}(y_2y_1),$$

we see that $M_2(K)$ satisfies the multilinear trace identity

$$f(y_1,y_2) = (y_1y_2+y_2y_1)-(\mathrm{tr}(y_1)y_2+\mathrm{tr}(y_2)y_1)+(\mathrm{tr}(y_1)\mathrm{tr}(y_2)-\mathrm{tr}(y_1y_2))e = 0.$$

Now we replace y_1 and y_2 respectively by $x_{\sigma(1)}x_{\sigma(2)}$ and $x_{\sigma(3)}x_{\sigma(4)}$ and take the alternating sum on $\sigma \in S_4$:

$$0 = \sum_{\sigma\in S_4}(\mathrm{sign}\sigma)f(x_{\sigma(1)}x_{\sigma(2)}, x_{\sigma(3)}x_{\sigma(4)}) =$$

$$= 2\sum_{\sigma\in S_4}(\mathrm{sign}\sigma)(x_{\sigma(1)}x_{\sigma(2)}x_{\sigma(3)}x_{\sigma(4)} - \mathrm{tr}(x_{\sigma(1)}x_{\sigma(2)})x_{\sigma(3)}x_{\sigma(4)})+$$

$$+\sum_{\sigma\in S_4}(\mathrm{sign}\sigma)(\mathrm{tr}(x_{\sigma(1)}x_{\sigma(2)})\mathrm{tr}(x_{\sigma(3)}x_{\sigma(4)}) - \mathrm{tr}(x_{\sigma(1)}x_{\sigma(2)}x_{\sigma(3)}x_{\sigma(4)})e).$$

The trace is invariant under cyclic permutations, hence

$$\mathrm{tr}(x_{\sigma(1)}x_{\sigma(2)}) = \mathrm{tr}(x_{\sigma(2)}x_{\sigma(1)}),$$

$$\mathrm{tr}(x_{\sigma(1)}x_{\sigma(2)}x_{\sigma(3)}x_{\sigma(4)}) = \mathrm{tr}(x_{\sigma(2)}x_{\sigma(3)}x_{\sigma(4)}x_{\sigma(1)}).$$

On the other hand, in each of the pairs

$$(\sigma(1),\sigma(2),\sigma(3),\sigma(4)) \text{ and } (\sigma(2),\sigma(1),\sigma(3),\sigma(4)),$$

$$(\sigma(1),\sigma(2),\sigma(3),\sigma(4)) \text{ and } (\sigma(2),\sigma(3),\sigma(4),\sigma(1))$$

the permutations are of different parity and the summands containing traces vanish in

$$\sum_{\sigma\in S_4}(\mathrm{sign}\sigma)f(x_{\sigma(1)}x_{\sigma(2)},x_{\sigma(3)}x_{\sigma(4)})=0.$$

Therefore, we obtain that

$$2s_4=2\sum_{\sigma\in S_4}(\mathrm{sign}\sigma)x_{\sigma(1)}x_{\sigma(2)}x_{\sigma(3)}x_{\sigma(4)}=0,$$

and this completes the proof.

Now we give the Rosset proof of the Amitsur-Levitzki theorem.

Lemma 7.1.7 *Let C be a commutative algebra over $\mathbb{Q}$ and let a be a $k\times k$ matrix with entries from C. If* $\mathrm{tr}(a^q)=0$ *for $q=1,2,\ldots,k$, then $a=0$.*

Proof. Let D be any field of characteristic 0 and let $a\in M_k(D)$. As in the Razmyslov proof of Amitsur-Levitzki Theorem 7.1.6,

$$a^k=\sum_{q=1}^{k}\alpha_q a^{k-q},$$

where α_q is a polynomial without constant term and with rational coefficients in $\mathrm{tr}(a^r)$, $r=1,2,\ldots,q$. In particular, the same formula holds if the entries of a are from a polynomial algebra $K[Y]$ for some set Y (embed $K[Y]$ into its field of fractions!). Now the algebra C is a homomorphic image of some polynomial algebra (use that the polynomial algebra is free in the variety of all commutative algebras!). Let $\phi:K[Y]\longrightarrow C$ be a homomorphism. Then ϕ induces a homomorphism $\tilde{\phi}:M_k(K[Y])\longrightarrow M_k(C)$ defined by $\tilde{\phi}(a_{ij})=(\phi(a_{ij}))$. For any $a\in M_k(K[Y])$

$$\mathrm{tr}(\tilde{\phi}a)=\phi(\mathrm{tr}(a)).$$

Therefore

$$a^k=\sum_{q=1}^{k}\alpha_q(\mathrm{tr}(a),\ldots,\mathrm{tr}(a^q))a^{k-q},\ a\in M_k(C).$$

If $\mathrm{tr}(a^q)=0$ for $q=1,2,\ldots,k$, then $a^k=0$.

The Rosset Proof of Amitsur-Levitzki Theorem 7.1.8 [230] Again, we consider $K=\mathbb{Q}$. Let E be the Grassmann algebra generated by $e_1,e_2,\ldots$ and let $E=E_0\oplus E_1$ be its canonical $\mathbb{Z}_2$-grading, i.e. E_0 and E_1 are spanned respectively by the products $e_{i_1}\ldots e_{i_l}$ of even and odd length. We have seen in Exercise 2.1.4 that E_0 is a commutative subalgebra of E. Let $r_1,\ldots,r_{2k}$ be matrices in $M_k(\mathbb{Q})$. Then

$$b = r_1e_1 + \ldots + r_{2k}e_{2k}$$

is a $k \times k$ matrix with entries from the (noncommutative) Grassmann algebra. Since $e_ie_j = -e_je_i$, we obtain that

$$a = b^2 = \sum_{i=1}^{2k}\sum_{j=1}^{2k} r_ir_je_ie_j = \sum_{i<j}(r_ir_j - r_jr_i)(e_ie_j)$$

and a is a matrix with entries from the commutative algebra E_0. It is easy to see (check it for $q = 2$) that

$$a^q = \sum s_{2q}(r_{i_1}, \ldots, r_{i_{2q}})e_{i_1} \ldots e_{i_{2q}},\ i_1 < \ldots < i_{2q},$$

$$\mathrm{tr}(a^q) = \sum \mathrm{tr}(s_{2q}(r_{i_1}, \ldots, r_{i_{2q}}))e_{i_1} \ldots e_{i_{2q}},\ i_1 < \ldots < i_{2q}.$$

As in the proof of Razmyslov we see that

$$\mathrm{tr}(s_{2q}(r_1, \ldots, r_{2q})) = 0,\ r_1, \ldots, r_{2q} \in M_k(\mathbb{Q}).$$

By Lemma 7.1.7

$$a^k = s_{2k}(r_1, \ldots, r_{2k})e_1 \ldots e_{2k} = 0,\ s_{2k}(r_1, \ldots, r_{2k}) = 0,$$

and this shows that the standard polynomial s_{2k} is an identity for $M_k(\mathbb{Q})$.

I think that now the reader is prepared to try to "solve" the following exercise.

Exercise 7.1.9 Read the original proof of the Amitsur-Levitzki theorem [11] and the proofs of Swan [249] and of Szigeti, Tuza and Révész [250].

The following exercise contains a result obtained independently by Chang [46] and Giambruno and Sehgal [114]. It answers a problem risen by Formanek.

Exercise 7.1.10 Answer in the affirmative the conjecture of Formanek that $n = 2k$ is the minimal positive integer such that the matrix algebra $M_k(K)$ ($\mathrm{char} K = 0$) satisfies the *double Capelli identity*

$$f(x_1, \ldots, x_n; y_1, \ldots, y_n) = \sum_{\sigma,\tau \in S_n} (\mathrm{sign}(\sigma\tau))x_{\sigma(1)}y_{\tau(1)} \ldots x_{\sigma(n)}y_{\tau(n)}.$$

Hint. (i) Consider the "double staircase"

$$\bar{x}_1 = e_{11}, \bar{y}_1 = e_{12}, \bar{x}_2 = e_{22}, \bar{y}_2 = e_{23}, \ldots, \bar{x}_k = e_{kk},$$

$$\bar{y}_k = e_{kk}, \bar{x}_{k+1} = e_{k,k-1}, \bar{y}_{k+1} = e_{k-1,k-1}, \ldots, \bar{x}_{2k-1} = e_{21}, \bar{y}_{2k-1} = e_{11}.$$

Using the graph theoretic interpretation of the matrix units and their products, show that $f(\bar{x}_1,\ldots,\bar{x}_{2k-1};\bar{y}_1,\ldots,\bar{y}_{2k-1})\neq 0$.

(ii) (We give the proof of Domokos [65]) Let $u_1,\ldots,u_{2k}$, $v_1,\ldots,v_{2k}$ be $k\times k$ matrices. Form the $2k\times 2k$ matrices

$$w_{2q-1}=\begin{pmatrix}0 & u_q\\ 0 & 0\end{pmatrix},\ w_{2q}=\begin{pmatrix}0 & 0\\ v_q & 0\end{pmatrix},\ q=1,\ldots,2k.$$

Since the product w_iw_j is not zero only if i and j are of different parity (why?), we obtain that

$$s_{4k}(w_1,\ldots,w_{4k})=\begin{pmatrix}f(u_1,\ldots,u_{2k};v_1,\ldots,v_{2k}) & 0\\ 0 & f(v_1,\ldots,v_{2k};u_1,\ldots,u_{2k})\end{pmatrix}$$

and this is 0 in virtue of the Amitsur-Levitzki theorem for $M_{2k}(K)$.

The proof of Chang [46] is combinatorial and contains the stronger result that the double Capelli identity in Exercise 7.1.10 is a consequence of the standard identity $s_{2k}=0$. The proof of Giambruno and Sehgal [114] is based on the Rosset proof of Amitsur-Levitzki Theorem 7.1.8. I hope that the reader will try to read these two papers. They are short, well written and one really enjoys reading them. Another proof can be found in the paper by Szigeti, Tuza and Révész [250] (see the comments after Amitsur-Levitzki Theorem 7.1.3).

One of the central problems in the theory of PI-algebras is the following.

Problem 7.1.11 Find a basis of the polynomial identities for the algebra of $k\times k$ matrices, $k\geq 2$, over a field of characteristic 0.

The complete answer is known for 2×2 matrices only. In 1973 Razmyslov [217] found a basis of 9 polynomial identities of degree 4, 5 and 6. Using this fact Drensky [72] showed in 1981 that the polynomial identities

$$s_4(x_1,x_2,x_3,x_4)=0,\ [[x_1,x_2]^2,x_1]=0$$

form a minimal basis of the polynomial identities of $M_2(K)$. The Hall identity in two variables $[[x_1,x_2]^2,x_1]=0$ is a consequence of the identity of algebraicity in Exercise 2.1.8. There was a hope that the standard identity $s_{2k}=0$ and the identity of algebraicity

$$\sum_{\sigma\in S_{k+1}}(\text{sign}\sigma)x^{\sigma(0)}y_1x^{\sigma(1)}y_2\ldots y_kx^{\sigma(k)}=0$$

form a basis for $T(M_k(K))$ also for $k>2$. Okhitin [198] constructed a polynomial identity of degree 9 for $M_3(K)$ which does not follow from these two identities. Domokos [66] found other new identities for 3×3 matrices. The most general known result about the identities of $k\times k$ matrices for any k is due to Razmyslov [219] and Procesi [214] who described the trace polynomial identities. Freely restated, their result gives that all polynomial identities for $M_k(K)$ follow from the Cayley-Hamilton theorem. We have already seen partial examples of this fact in the Razmyslov proof of Amitsur-Levitzki Theorem 7.1.6 and in the proof that the identity of algebraicity holds for the matrix algebra (Exercise 2.1.8).

In the next several exercises and problems we assume that $\text{char} K = 0$.

Exercise 7.1.12 Show that the identity of algebraicity for $M_k(K)$, $k > 1$, does not follow from the standard identity $s_{2k} = 0$.

Hint. Let

$$a_k(x, y_1, \ldots, y_k) = \sum_{\sigma \in S_{k+1}} (\text{sign}\sigma) x^{\sigma(0)} y_1 x^{\sigma(1)} y_2 \ldots y_k x^{\sigma(k)} = 0$$

be the identity of algebraicity. If it follows from the standard identity $s_{2k} = 0$, then

$$b(x, y) = a_k(x, y, \ldots, y) = \sum \alpha_i u_i s_{2k}(v_{i_1}, \ldots, v_{i_{2k}}) w_i, \ \alpha_i \in K,$$

where u_i, v_{i_j}, w_i are monomials in $K\langle x, y\rangle$. We may assume that the total degree of u_i, v_{i_j}, w_i in y is k and in x is $\frac{1}{2}k(k+1)$. This means that in the standard identity $s_{2k}(v_{i_1}, \ldots, v_{i_{2k}})$ not more than k monomials v_{i_j} contain y and the others are positive (why? – $s_{2k}(x_1, \ldots, x_{2k})$ is proper!) powers of x. Since the standard identity is skew symmetric, we obtain that all monomials v_{i_j} are different and this gives that the total degree of x is not less than $1 + 2 + \ldots + k$. Derive from here that all monomials v_j containing y should be equal to y and, since $k \geq 2$, this is impossible.

Exercise 7.1.13 Show that for k sufficiently large, $M_k(K)$ has polynomial identities in two variables which are consequences of the standard identity and are of degree less than the degree of the identity of algebraicity (with $y_1 = \ldots = y_k = y$).

Hint. Order all monomials in two variables in $K\langle x, y\rangle$ by ascending degree, e.g.

$$u_1 = x,\ u_2 = y,\ u_3 = x^2,\ u_4 = xy,\ u_5 = yx,\ u_6 = y^2, \ldots$$

and, using "economically" these elements, try to construct a nonzero polynomial in $K\langle x, y\rangle$ of the form

$$s_{2k}(u_{i_1}, \ldots, u_{i_{2k}}),\ i_1 < \ldots < i_{2k},$$

Try e.g. with $s_{2k}(x, y, x^2, xy, yx, y^2, \ldots)$ or with

$$s_{2k}(x, y, xy, yx, x^2y, xyx, yx^2, x^3y, x^2yx, xyx^2, yx^3, \ldots)$$

Show that for k sufficiently large, the degree of the obtained identity is less than $\frac{1}{2}k(k+1)+k$, which is the degree of the identity of algebraicity. If there are some difficulties, see Exercise 7.4.8.

Problem 7.1.14 Find the minimal degree of the polynomial identities in two variables for the matrix algebra $M_k(K)$, $k > 3$.

Problem 7.1.15 Find the minimal degree of the polynomial identities for $M_k(K)$ which do not follow from the standard identity $s_{2k} = 0$, $k > 3$.

Leron [171] proved that for $k > 2$ all polynomial identities for $M_k(K)$ of degree $2k+1$ follow from the standard identity. For $k = 3$ the same is true also for the polynomial identities of degree $8 = 2.3 + 2$ (Drensky and Azniv Kasparian [91]). Since the identity of algebraicity is of degree 9, the minimal degree of the polynomial identities for $M_3(K)$ which do not follow from $s_6 = 0$ is equal to 9.

7.2 Generic Matrices

In this section we assume that the base field K is arbitrary and for the integer $k \geq 2$ we fix the notation $\Omega = \Omega_k$ for the K-algebra of the polynomials in infinitely many commuting variables

$$\Omega_k = K[y_{pq}^{(i)} \mid p,q = 1,\dots k,\ i = 1,2,\dots].$$

Definition 7.2.1 The $k \times k$ matrices with entries from Ω_k

$$y_i = \sum_{p,q=1}^{k} y_{pq}^{(i)} e_{pq},\ i = 1,2,\dots,$$

are called *generic $k \times k$ matrices.* The algebra R_k generated by the generic $k \times k$ matrices is the *generic $k \times k$ matrix algebra.* We denote by R_{km} the subalgebra of R_k generated by the first m generic matrices $y_1,\dots,y_m$.

For example, for $k = m = 2$, changing the notation and assuming that $y_{pq}^{(1)} = x_{pq}$, $y_{pq}^{(2)} = y_{pq}$, the algebra R_{22} is generated by

$$x = \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix},\ y = \begin{pmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{pmatrix}.$$

For any commutative K-algebra C, the $k \times k$ matrices with entries from C can be obtained by specializations of the generic matrices, e.g.

$$a = \sum_{p,q=1}^{k} \gamma_{pq} e_{pq},\ \gamma_{pq} \in C,$$

is obtained from

$$y_1 = \sum_{p,q=1}^{k} y_{pq}^{(1)} e_{pq}$$

by replacing the variables $y_{pq}^{(1)}$ with γ_{pq}.

Exercise 7.2.2 If the base field is infinite, show that the generic matrix algebra R_k is isomorphic to the relatively free algebra $F(M_k(K))$ of the $k \times k$ matrix variety, i.e.

$$R_k \cong K\langle X\rangle / T(M_k(K)).$$

If K is a finite field and P is any infinite extension of K, show that $R_k \cong F(M_k(P))$.

Hint. For any infinite field P containing K, show that the kernels of the canonical homomorphisms

$$K\langle X\rangle \longrightarrow F(M_k(P)),\ K\langle X\rangle \longrightarrow R_k$$

coincide, i.e. $f(x_1, \ldots, x_n) \in K\langle X\rangle$ is a polynomial identity for $M_k(P)$ if and only if $f(y_1, \ldots, y_n) = 0$ in R_k.

Exercise 7.2.3 Let $P = K[\xi]$ be an extension of degree k of the base field K (where ξ is a primitive element). Considering P as a k-dimensional vector space over K, let $\{\eta_1, \ldots, \eta_k\}$ be a basis of P. Let $\bar{\xi}$ be the linear operator of P defined by $\bar{\xi}(\alpha) = \xi\alpha$, $\alpha \in P$. Show that, up to a multiplicative constant, the characteristic polynomial of $\bar{\xi}$ coincides with the minimal polynomial of ξ over K.

Hint. If $f(\xi) = 0$, $f(t) \in K[t]$, then $f(\bar{\xi}) = 0$. Use the Cayley-Hamilton theorem and the fact that the minimal polynomial of ξ is of degree k and is irreducible over K.

Exercise 7.2.4 Calculate the characteristic polynomial

$$f_a(\lambda) = \det(a - \lambda e)$$

of the matrix

$$a = \begin{pmatrix} 0 & 0 & \ldots & 0 & 0 & \alpha_0 \\ 1 & 0 & \ldots & 0 & 0 & \alpha_1 \\ 0 & 1 & \ldots & 0 & 0 & \alpha_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \ldots & 1 & 0 & \alpha_{k-2} \\ 0 & 0 & \ldots & 0 & 1 & \alpha_{k-1} \end{pmatrix}.$$

Answer.

$$f_a(\lambda) = (-1)^k(\lambda^k - (\alpha_{k-1}\lambda^{k-1} + \alpha_{k-2}\lambda^{k-2} + \ldots + \alpha_1\lambda + \alpha_0)).$$

Lemma 7.2.5 *The eigenvalues of the $k \times k$ generic matrix y_1 are pairwise different.*

Proof. We consider y_1 as a matrix with entries from the field of fractions of the polynomial algebra Ω_k. Let $f(\lambda)$ be the characteristic polynomial of y_1. Let $f(\lambda)$ have multiple zeros. Then the discriminant of $f(\lambda)$ is equal to zero. Since every $k \times k$ matrix a with entries from K is a specialization of y_1, the discriminant of the characteristic polynomial $f_a(\lambda)$ is also equal to 0 and this means that a has multiple eigenvalues. We shall prove the lemma if we find a matrix $a \in M_k(K)$ without multiple eigenvalues. If K is infinite we can do this in an obvious way (how?). If $K = \mathbb{F}_q$, then there exists an irreducible over $\mathbb{F}_q$ polynomial $g(\lambda) \in \mathbb{F}_q[\lambda]$ of degree k and we may use Exercise 7.2.4 in order to construct a. Another possibility is to use Exercise 7.2.3 for $P = \mathbb{F}_{q^k}$, because the minimal polynomial of a primitive element of P has no multiple zeros.

Corollary 7.2.6 *Let Ω_k be as above and let*

$$y'_1 = \sum_{p=1}^{k} y_{pp}^{(1)} e_{pp},\ y'_i = y_i,\ i > 1.$$

The algebra R'_k generated by $y'_1, y'_2, y'_3, \ldots$ is isomorphic to the generic matrix algebra R_k.

Proof. Let Ξ be the algebraic closure of the field of fractions of the polynomial algebra Ω_k. By Lemma 7.2.5, the generic matrix y_1 has no multiple eigenvalues. Hence there exists an invertible matrix z with entries from Ξ such that the matrix $u_1 = z^{-1}y_1z$ is diagonal. Let

$$u_i = z^{-1}y_iz,\ i = 1, 2, \ldots$$

Denote by U_k the K-subalgebra of $M_k(\Xi)$ generated by $u_1, u_2, \ldots$. Clearly U_k is isomorphic to R_k. Let $\phi : R_k \longrightarrow R'_k$ and $\psi : R'_k \longrightarrow U_k$ be the K-algebra homomorphisms extending respectively the maps $\phi_0 : y_i \to y'_i$ and $\psi_0 : y'_i \to u_i$, $i = 1, 2, \ldots$. Since the matrices u_i are obtained as specializations of the "generic" matrices y'_i (in the class of all sequences of matrices, first of which is diagonal), ψ *is* a homomorphism. The composition $\psi\phi : R_k \longrightarrow U_k$ is the isomorphism defined by $y_i \to u_i = z^{-1}y_iz$. This implies that $\mathrm{Ker}\phi = 0$ and, since ϕ is onto R'_k, it is an isomorphism.

Corollary 7.2.6 allows to replace one of the generic matrices by a diagonal generic matrix which will be very useful in the further considerations. Sometimes, if we consider a single generic $k \times k$ matrix y, we shall use Greek characters for the diagonal entries of y and shall write e.g. $y=\sum_{p=1}^{k}\eta_p e_{pp}$ instead of $y=\sum_{p=1}^{k}y_{pp}e_{pp}$, assuming that $\eta_1,\ldots,\eta_k$ are commuting variables.

Exercise 7.2.7 Show that we may assume that in R_k the first generic matrix is diagonal and the second has the same first row and column, e.g.

$$y_1=\sum_{p=1}^{k}y_{pp}^{(1)}e_{pp},\, y_2=\sum_{p,q=1}^{k}y_{pq}^{(2)}e_{pq},$$

and $y_{1q}^{(2)}=y_{q1}^{(2)}$, $q=2,3,\ldots,k$.

Hint. Conjugate the algebra R_k' in Corollary 7.2.6 with a suitable invertible diagonal matrix $z=\sum_{p=1}^{k}\xi_p e_{pp}\in M_k(\Xi)$. Then apply the arguments of the proof of the corollary.

Exercise 7.2.8 Let A be a finite dimensional algebra with basis $\{a_1,\ldots,a_m\}$ over an infinite field K and let

$$\Omega_A=K[z_p^{(i)}\mid p=1,\ldots,m,\; i=1,2,\ldots]$$

be the polynomial algebra. Show that the "generic" algebra R_A generated as a K-algebra by the elements

$$z_i=\sum_{p=1}^{m}z_p^{(i)}a_p,\; i=1,2,\ldots,$$

is isomorphic to the relatively free algebra $F(A)$. (In order to define the multiplication in R_A, we assume that the polynomials of Ω_A commute with the basis elements of A and the multiplication between $a_1,\ldots,a_m$ is as in A. In other words, we consider R_A naturally embedded into the tensor product of K-algebras $\Omega_A\otimes A$.)

Hint. Apply the same arguments as for the generic matrices in Exercise 7.2.2.

7.3 Central Polynomials

Definition 7.3.1 Let R be an algebra. The polynomial $c(x_1,\ldots,x_n)\in K\langle X\rangle$ is called a *central polynomial* for R if $c(x_1,\ldots,x_n)$ has no constant term, $c(r_1,\ldots,r_n)$ belongs to the centre of R for all $r_1,\ldots,r_n\in R$, and $c(x_1,\ldots,x_n)=0$ is not a polynomial identity for R.

Exercise 7.3.2 Show that the polynomial

$$c(x_1,x_2)=[x_1,x_2]^2$$

is central for the matrix algebra $M_2(K)$.

Hint. Use the Cayley-Hamilton theorem and the equality $\mathrm{tr}([x_1,x_2])=0$ (see Exercise 2.1.7 (ii)). Prove that over any (also finite) field there exist matrices $a_1,a_2\in M_2(K)$ with $\det([a_1,a_2])\neq 0$.

In 1956 Kaplansky [133] gave a talk where he asked several problems which motivated significant research activity in the next decades. One of his problems was whether there exists a central polynomial for the matrix algebras $M_k(K)$ for $k>2$. A revised version [134] of the problems of Kaplansky appeared in 1970.

Problem 7.3.3 (Kaplansky) Does there exist a multihomogeneous central polynomial for the matrix algebra $M_k(K)$, $k>2$?

The answer to the problem of Kaplansky was given in 1972-1973 by Formanek [104] and Razmyslov [218] and this was very important for ring theory. We refer to the books by Jacobson [128], Rowen [231] and Formanek [108] to see how some important theorems were established or simplified using central polynomials. There are two essentially different methods for construction of central polynomials due to Formanek and Razmyslov. We shall give both of them. Later, an alternative proof of the existence of central polynomials based on results of Amitsur was given by Kharchenko [145]. Combining ideas of Formanek and Razmyslov with other methods, various central polynomials were also constructed by Halpin [124], Drensky [85], Giambruno and Angela Valenti [115]. We start with the central polynomial of Formanek.

Let $K[u_1,\ldots,u_{k+1}]$ be the polynomial algebra in $k+1$ commuting variables. We define a linear mapping θ (not an algebra homomorphism) from $K[u_1,\ldots,u_{k+1}]$ to the free algebra $K\langle x,y_1,\ldots,y_k\rangle$ of rank $k+1$ in the following way. If

$$g(u_1,\ldots,u_{k+1})=\sum\alpha_a u_1^{a_1}\ldots u_{k+1}^{a_{k+1}},\ \alpha_a\in K,$$

then

$$\theta(g)(x,y_1,\ldots,y_k)=\sum\alpha_a x^{a_1}y_1x^{a_2}y_2x^{a_3}y_3\ldots x^{a_k}y_kx^{a_{k+1}}.$$

Exercise 7.3.4 Show that for

$$\bar{x}=\sum_{p=1}^{k}\rho_pe_{pp},\ \bar{y}_q=e_{i_qj_q}\in M_k(K),\ \rho_p\in K,\ p,q=1,\ldots,k,$$

and for $g(u_1,\ldots,u_{k+1})\in K[u_1,\ldots,u_{k+1}]$,

$$\theta(g)(\bar{x}, \bar{y}_1, \ldots, \bar{y}_k) = g(\rho_{i_1}, \rho_{i_2}, \ldots, \rho_{i_k}, \rho_{j_k}) e_{i_1 j_1} \ldots e_{i_k j_k}.$$

Hint. Use that $\bar{x} e_{ij} = \rho_i e_{ij}$, $e_{ij} \bar{x} = \rho_j e_{ij}$.

Theorem 7.3.5 (Formanek [104]) *Let*

$$g(u_1, \ldots, u_{k+1}) = \prod_{2 \leq p \leq k} (u_1 - u_p)(u_{k+1} - u_p) \prod_{2 \leq p < q \leq k} (u_p - u_q)^2.$$

The polynomial of the free associative algebra $K\langle x, y_1, \ldots, y_k \rangle$

$$c(x, y_1, \ldots, y_k) = \theta(g)(x, y_1, \ldots, y_k) + \theta(g)(x, y_2, \ldots, y_k, y_1) +$$
$$+ \ldots + \theta(g)(x, y_k, y_1, \ldots, y_{k-1})$$

is a central polynomial for the matrix algebra $M_k(K)$ *over every field* K.

Proof. It is sufficient to show that the polynomial $c(\bar{x}, \bar{y}_1, \ldots, \bar{y}_k)$ is a scalar matrix when $\bar{x}, \bar{y}_1, \ldots, \bar{y}_k$ are generic matrices (why?) and $c(x, y_1, \ldots, y_k) = 0$ is not a polynomial identity for $M_k(K)$. By Corollary 7.2.6, we may assume that the generic matrix $\bar{x}$ is diagonal,

$$\bar{x} = \sum_{p=1}^{k} \xi_p e_{pp},$$

where $\xi_1, \ldots, \xi_k$ are commuting variables. Since $c(x, y_1, \ldots, y_k)$ is multilinear in $y_1, \ldots, y_k$, we may also assume that $\bar{y}_1, \ldots, \bar{y}_k$ are matrix units. Hence

$$c(\bar{x}, \bar{y}_1, \ldots, \bar{y}_k) = \sum_{q=1}^{k} \theta(\bar{x}, e_{i_q j_q}, \ldots, e_{i_k j_k}, e_{i_1 j_1}, \ldots, e_{i_{q-1} j_{q-1}}) =$$

$$= \sum_{q=1}^{k} g(\xi_{i_q}, \ldots, \xi_{i_k}, \xi_{i_1}, \ldots, \xi_{i_{q-1}}, \xi_{j_{q-1}}) e_{i_q j_q} \ldots e_{i_k j_k} e_{i_1 j_1} \ldots e_{i_{q-1} j_{q-1}}.$$

Since we have chosen g in a very special way,

$$g(\xi_{i_q}, \ldots, \xi_{i_k}, \xi_{i_1}, \ldots, \xi_{i_{q-1}}, \xi_{j_{q-1}}) = 0$$

if some of the variables ξ_{i_p} and $\xi_{i_{p'}}$ are equal or if $\xi_{j_{q-1}}$ is equal to some ξ_{i_p} with $p \neq q$. On the other hand, if $i_1, \ldots, i_k$ is a permutation of $1, \ldots, k$ and $i_q = j_{q-1}$, then

$$g(\xi_{i_q}, \ldots, \xi_{i_k}, \xi_{i_1}, \ldots, \xi_{i_{q-1}}, \xi_{i_q}) =$$
$$= g(\xi_1, \xi_2, \ldots, \xi_k, \xi_1) = \prod_{1 \leq p < p' \leq k} (\xi_p - \xi_{p'})^2,$$

$$\theta(g)(\bar{x}, e_{i_q i_{q+1}}, e_{i_{q+1} i_{q+2}} \ldots, e_{i_k i_1}, e_{i_1 i_2}, \ldots, e_{i_{q-1} i_q}) = \prod_{1 \leq p < p' \leq k} (\xi_p - \xi_{p'})^2 e_{i_q i_q}.$$

Therefore,

$$c(\bar{x}, e_{i_1i_2}, e_{i_2i_3}, \dots, e_{i_{k-1}i_k}, e_{i_ki_1}) =$$

$$= \prod_{1\leq p<q\leq k} (\xi_p - \xi_q)^2 \sum_{q=1}^{k} e_{i_qi_q} = \prod_{1\leq p<q\leq k} (\xi_p - \xi_q)^2 e,$$

if $i_1, i_2, \dots, i_k$ is a permutation of $1, 2, \dots, k$ and

$$c(\bar{x}, e_{i_1j_1}, e_{i_2j_2}, \dots, e_{i_kj_k}) = 0$$

otherwise. Now, the only statement to prove is that $c(x, y_1, \dots, y_k) = 0$ is not a polynomial identity for $M_k(K)$. If $\bar{x}$ is a diagonal matrix with different eigenvalues $\rho_1, \dots, \rho_k$ we see that

$$c(\bar{x}, e_{12}, e_{23}, \dots, e_{k-1,k}, e_{k1}) = \prod_{1\leq p<q\leq k} (\rho_p - \rho_q)^2 e$$

which is a nonzero scalar matrix in $M_k(K)$. If $K = \mathbb{F}_q$, we consider a matrix $\bar{x} \in M_k(\mathbb{F}_q)$ with different eigenvalues. Then $\bar{x}$ is diagonalizable over some $\mathbb{F}_{q^m}$. Hence $c(\bar{x}, \bar{y}_1, \dots, \bar{y}_k) \neq 0$ in $M_k(\mathbb{F}_{q^m})$. Since $c(\bar{x}, \bar{y}_1, \dots, \bar{y}_k)$ is a linear combination (with coefficients in $\mathbb{F}_{q^m}$) of $c(\bar{x}, e_{i_1j_1}, \dots, e_{i_kj_k}) \in M_k(\mathbb{F}_q)$, some $c(\bar{x}, e_{i_1j_1}, \dots, e_{i_kj_k})$ is different from 0. This completes the proof of the theorem of Formanek.

Now we give the Razmyslov approach to central polynomials. It uses the notion of weak polynomial identities [217, 218] and of the Razmyslov transform [218]. See the book by Razmyslov [221] for account on his methods containing also applications to many other problems of the theory of PI-algebras, Lie algebras with polynomial identities and groups with identical relations.

Definition 7.3.6 The polynomial $f(x_1, \dots, x_n) \in K\langle X\rangle$ is called a *weak polynomial identity* for the matrix algebra $M_k(K)$ if $f(a_1, \dots, a_n) = 0$ for all traceless matrices $a_1, \dots, a_n$ in $M_k(K)$ (i.e. $a_1, \dots, a_n$ belong to the Lie subalgebra $sl_k(K)$ of $M_k(K)^{(-)}$). The weak polynomial identity is *essential* if it is not a polynomial identity for $M_k(K)$.

Exercise 7.3.7 Show that $[x_1^2, x_2] = 0$ is an essential weak polynomial identity for $M_2(K)$.

Hint. If $\mathrm{tr}(a_1) = 0$ then the Cayley-Hamilton theorem gives that a_1^2 is a scalar matrix. Find matrices $a_1, a_2 \in M_2(K)$, $\mathrm{tr}(a_1) \neq 0$, such that $[a_1^2, a_2] \neq 0$.

Exercise 7.3.8 If $f(x_1, \dots, x_n) = 0$ is a weak polynomial identity for $M_k(K)$, then $f([x_1, x_{n+1}], [x_2, x_{n+2}], \dots, [x_n, x_{2n}]) = 0$ is an ordinary polynomial identity.

Hint. $\mathrm{tr}([a,b]) = 0$ for any $a, b \in M_k(K)$.

Exercise 7.3.9 Show that the Capelli polynomial

$$d_{k^2}(x_1, \ldots, x_{k^2}; y_1, \ldots, y_{k^2+1})$$

in k^2 skew symmetric variables is an essential weak polynomial identity for $M_k(K)$ which vanishes when the x's are replaced by elements of $sl_k(K)$ and the y's are arbitrary matrices of $M_k(K)$.

Hint. Since $\dim sl_k(K) = k^2 - 1$, the polynomial d_{k^2} vanishes when the x's are replaced by elements of $sl_k(K)$. On the other hand, d_{k^2} is not zero on $M_k(K)$ (see Exercise 7.1.2).

Exercise 7.3.10 (Halpin [124]) Let the base field K be infinite and let the symmetric group S_k act on the set $\{0, 1, \ldots, k-2, k\}$. Show that the polynomial

$$w(x, y_1, \ldots, y_{k-1}) = \sum_{\sigma \in S_k} (\mathrm{sign}\sigma) x^{\sigma(0)} y_1 x^{\sigma(1)} y_2 \ldots y_{k-2} x^{\sigma(k-2)} y_{k-1} x^{\sigma(k)}$$

is an essential weak polynomial identity for $M_k(K)$ which vanishes for $x \in sl_k(K)$ and $y_1, \ldots, y_{k-1} \in M_k(K)$.

Hint. The Cayley-Hamilton theorem gives that

$$a^k - \mathrm{tr}(a)a^{k-1} + \ldots + (-1)^k \det(a) e = 0,\ a \in M_k(K).$$

If $\mathrm{tr}(a) = 0$, then $1, a, a^2, \ldots, a^{k-2}, a^k$ are linearly dependent and

$$w(a, b_1, \ldots, b_{k-1}) = 0,\ b_1, \ldots, b_{k-1} \in M_k(K).$$

If $\bar{x} = \sum_{p=1}^{k} \rho_p e_{pp}$, show that

$$w(\bar{x}, e_{12}, e_{23}, \ldots, e_{k-1,k}) = \Delta(\rho_1, \ldots, \rho_k) e_{1k},$$

where

$$\Delta(\rho_1, \ldots, \rho_k) = \begin{vmatrix} 1 & 1 & 1 & \ldots & 1 & 1 \\ \rho_1 & \rho_2 & \rho_3 & \ldots & \rho_{k-1} & \rho_k \\ \rho_1^2 & \rho_2^2 & \rho_3^2 & \ldots & \rho_{k-1}^2 & \rho_k^2 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \rho_1^{k-2} & \rho_2^{k-2} & \rho_3^{k-2} & \ldots & \rho_{k-1}^{k-2} & \rho_k^{k-2} \\ \rho_1^k & \rho_2^k & \rho_3^k & \ldots & \rho_{k-1}^k & \rho_k^k \end{vmatrix} =$$

$$= (\rho_1 + \ldots + \rho_k) \prod_{1 \leq p < q \leq k} (\rho_q - \rho_p).$$

Remark 7.3.11 The notion of weak polynomial identity was introduced by Razmyslov for several purposes. One of them was to construct central polynomials. Another purpose was to study the ordinary polynomial identities for matrices. For example, one of the main steps of the Razmyslov result [217] that the polynomial identities of the 2×2 matrix algebra over a field of characteristic 0 follow from those of degree ≤ 6 is that the weak polynomial identities of $M_2(K)$ are consequences (in sense similar to that for ordinary polynomial identities) of the weak identity $[x_1^2, x_2] = 0$. The notion of weak polynomial identity was further generalized to the case of representations of Lie (and other) algebras. We restrict our considerations in this direction to the definition only.

Definition 7.3.12 Let G be a Lie algebra with universal enveloping algebra $U(G)$ and let

$$\phi : U(G) \longrightarrow \mathrm{End} V$$

be an associative algebra homomorphism of $U(G)$ to the algebra of linear operators of a vector space V. Then ϕ is called a *representation* of G. A polynomial $f(x_1, \ldots, x_n) \in K\langle X\rangle$ is an *identity* for the representation ϕ if

$$f(\phi(g_1), \ldots, \phi(g_n)) = 0$$

for all $g_1, \ldots, g_n \in G$.

If we consider the Lie algebra $sl_k(K)$ with its natural k-dimensional representation (as $k \times k$ matrices with trace zero), we obtain the notion of weak polynomial identity. For further details on identities of representations and their applications one may see the books by Razmyslov [221] and Vovsi [265].

Exercise 7.3.13 Let $a_i, b_i \in M_k(K)$, $i = 1, \ldots, n$, and let

$$f(u) = \sum_{i=1}^{n} a_i u b_i = 0$$

for all matrices $u \in M_k(K)$. Show that

$$f^*(u) = \sum_{i=1}^{n} b_i u a_i = 0$$

for all $u \in M_k(K)$.

Hint. Let

$$a_i = \sum_{p,q=1}^{k} \alpha_{pq}^{(i)} e_{pq},\ b_i = \sum_{r,s=1}^{k} \beta_{rs}^{(i)} e_{rs},\ \alpha_{pq}^{(i)}, \beta_{rs}^{(i)} \in K.$$

Then

$$f(e_{qr}) = \sum_{p,s=1}^{k} \left(\sum_{i=1}^{n} \alpha_{pq}^{(i)} \beta_{rs}^{(i)} \right) e_{ps} = 0,$$

and for any fixed $p, q, r, s = 1, \ldots, k$,

$$\sum_{i=1}^{n} \alpha_{pq}^{(i)} \beta_{rs}^{(i)} = 0.$$

This implies that $f^*(e_{sp}) = 0$, $s, p = 1, \ldots, k$, and hence $f^*(u) = 0$ for any $u \in M_k(K)$.

Exercise 7.3.14 Define a symmetric bilinear form on the vector space $V = M_k(K)$ by

$$\langle a, b \rangle = \mathrm{tr}(ab),\ a, b \in M_k(K).$$

(i) Show that this form is nondegenerate, i.e. $\langle u, V \rangle = 0$ implies $u = 0$.
(ii) Derive from (i) a new proof of Exercise 7.3.13.

Hint. (i) Multiply u by a suitable matrix $a \in M_k(K)$ in order to obtain $\mathrm{tr}(ua) \neq 0$.

(ii) The equality $f(u) = \sum_{i=1}^{n} a_i u b_i = 0$ for all matrices $u \in M_k(K)$ is equivalent to $\mathrm{tr}(f(u)v) = 0$ for all $u, v \in M_k(K)$. Use that

$$\mathrm{tr}(f(u)v) = \sum_{i=1}^{n} \mathrm{tr}(a_i u b_i v) = \sum_{i=1}^{n} \mathrm{tr}(b_i v a_i u) = \mathrm{tr}(f^*(v)u) = 0$$

for all $u, v \in M_k(K)$ implies that $f^*(v) = 0$ for all $v \in M_k(K)$.

Exercise 7.3.15 Show that any matrix with trace equal to zero in $M_k(K)$ is a linear combination of commutators.

Hint. Show this for the basis of $sl_k(K)$ consisting of e_{ij}, $i \neq j$, and $e_{11} - e_{ii}$, $i = 2, \ldots, k$.

Definition 7.3.16 Let the polynomial $f(x, y_1, \ldots, y_n) \in K\langle x, y_1, \ldots, y_n \rangle$ be linear (i.e. homogeneous of degree 1) in the variable x. Writing f in the form $f = \sum g_i x h_i$, where $g_i, h_i \in K\langle y_1, \ldots, y_n \rangle$, the *Razmyslov transform* of f is the polynomial

$$f^*(x, y_1, \ldots, y_n) = \sum h_i x g_i.$$

For example, if

$$f(x, y_1, y_2) = [xy_1 + y_1 x, y_2] = 1 \cdot x \cdot y_1 y_2 + y_1 \cdot x \cdot y_2 - y_2 \cdot x \cdot y_1 - y_2 y_1 \cdot x \cdot 1,$$

then

$f^*(x,y_1,y_2) = y_1y_2\cdot x\cdot 1 + y_2\cdot x\cdot y_1 - y_1\cdot x\cdot y_2 - 1\cdot x\cdot y_2y_1 = [y_2,x]y_1 + y_1[y_2,x]$.

Lemma of Razmyslov 7.3.17 *Let the polynomial*

$$f(x,y_1,\ldots,y_n)\in K\langle x,y_1,\ldots,y_n\rangle$$

be homogeneous of first degree in x *and let* $f^*(x,y_1,\ldots,y_n)$ *be the polynomial obtained by the Razmyslov transform. Then:*

(i) $f(x,y_1,\ldots,y_n)$ *is a polynomial identity for the matrix algebra* $M_k(K)$ *if and only if* $f^*(x,y_1,\ldots,y_n)$ *is a polynomial identity.*

(ii) $f^*(x,y_1,\ldots,y_n)$ *is a central polynomial for the algebra* $M_k(K)$ *if and only if* $f(x,y_1,\ldots,y_n)$ *is an essential weak polynomial identity and* $f([x,y_0],y_1,\ldots,y_n)$ *is a polynomial identity for* $M_k(K)$.

Proof. (i) Let

$$f(x,y_1,\ldots,y_n) = \sum_{i=1}^m a_i(y_1,\ldots,y_n)xb_i(y_1,\ldots,y_n)$$

and let, for fixed $r_1,\ldots,r_n\in M_k(K)$,

$$f(u) = f(u,r_1,\ldots,r_n) = \sum_{i=1}^m a_i(r_1,\ldots,r_n)ub_i(r_1,\ldots,r_n),\ u\in M_k(K).$$

By Exercise 7.3.13, $f(u)=0$ for all $u\in M_k(K)$ if and only if $f^*(u)=0$ and this gives the statement of (i).

(ii) Assuming again that

$$f(x,y_1,\ldots,y_n) = \sum_{i=1}^m a_i(y_1,\ldots,y_n)xb_i(y_1,\ldots,y_n)$$

we apply the Razmyslov transform to

$$g(x,y_0,y_1,\ldots,y_n) = f([x,y_0],y_1,\ldots,y_n) = \sum_{i=1}^m a_i(xy_0-y_0x)b_i$$

and obtain

$$g^*(x,y_0,y_1,\ldots,y_n) = \sum_{i=1}^m (y_0b_ixa_i - b_ixa_iy_0) =$$

$$= [y_0,\sum_{i=1}^m b_ixa_i] = [y_0,f^*(x,y_1,\ldots,y_n)].$$

Obviously, $f^*(x,y_1,\ldots,y_n)$ is a central polynomial (maybe vanishing) for $M_k(K)$ if and only if

$$g^*(x, y_0, y_1, \ldots, y_n) = [y_0, f^*(x, y_1, \ldots, y_n)]$$

is a polynomial identity and this means that $f([x, y_0], y_1, \ldots, y_n) = 0$ is a polynomial identity. Clearly $f^*(x, y_1, \ldots, y_n)$ does not vanish on $M_k(K)$ if and only if $f(x, y_1, \ldots, y_n)$ is not a polynomial identity. In order to finish the proof it is sufficient to use Exercise 7.3.15 that any matrix with trace zero is a linear combination of commutators which means that $f(x, y_1, \ldots, y_n) = 0$ is a weak polynomial identity.

Theorem 7.3.18 (Razmyslov [218]) *Let*

$$f = f(x, z_1, \ldots, z_{2k^2-2}, y_1, \ldots, y_{k^2-1}) =$$

$$= d_{k^2}(x, [z_1, z_2], \ldots, [z_{2k^2-3}, z_{2k^2-2}]; 1, y_1, \ldots, y_{k^2-1}, 1)$$

where $d_n(x_1, \ldots, x_n; y_1, \ldots, y_{n+1})$ is the Capelli polynomial. The Razmyslov transform applied to f gives a multilinear central polynomial for $M_k(K)$ over any field K.

Proof. By Exercise 7.3.9, $f(x, z_1, \ldots, y_{k^2-1}) = 0$ is a weak polynomial identity because it vanishes if we replace x by a traceless matrix (the matrices $[z_{2i-1}, z_{2i}]$ are already traceless). It is also clear that the replacement of x by a commutator gives rise to a polynomial identity. By Lemma of Razmyslov 7.3.17, it is sufficient to show that $f(x, z_1, \ldots, y_{k^2-1}) = 0$ is not a polynomial identity. We choose $2k^2-2$ traceless matrices $\bar{z}_i \in M_k(K)$ such that the set of commutators $[\bar{z}_{2i-1}, \bar{z}_{2i}]$, $i = 1, \ldots, k^2-1$, coincides with the basis of $sl_k(K)$ consisting of all e_{ij}, $i \neq j$, and $e_{ii} - e_{11}$, $i \neq 1$. Replacing x by $\bar{x} = e_{11}$ and using the skew symmetry of the Capelli identity we obtain that

$$f(\bar{x}, \bar{z}_1, \ldots, \bar{z}_{2k^2-3}, \bar{z}_{2k^2-2}, y_1, \ldots, y_{k^2-1}) = \pm d_{k^2}(e_{ij}; 1, y_1, \ldots, y_{k^2-1}, 1),$$

where $d_{k^2}(e_{ij}; 1, y_1, \ldots, y_{k^2-1}, 1)$ means that we have replaced the skew symmetric variables by the k^2 different matrix units. Now, as in Exercise 7.1.2, we may choose matrix units $\bar{y}_i$ to obtain a nonzero value of the Capelli identity. By Lemma of Razmyslov 7.3.17 the polynomial f^* is a central polynomial for $M_k(K)$.

Using the method of Razmyslov, Halpin [124] constructed another central polynomial. We give a simplified version of his theorem.

Exercise 7.3.19 (Halpin) Let $\mathrm{char} K = 0$, let S_k act on $\{0, 1, 2, \ldots, k-2, k\}$ and let

$$w_1(x, z_1, \ldots, z_m, y_1, \ldots, y_{k-1}),\ m = \frac{1}{2}k(k-1),$$

be the complete linearization of

$$w(x, y_1, \ldots, y_{k-1}) = \sum_{\sigma \in S_k} (\mathrm{sign}\sigma) x^{\sigma(0)} y_1 x^{\sigma(1)} y_2 \ldots y_{k-2} x^{\sigma(k-2)} y_{k-1} x^{\sigma(k)}.$$

(i) Let $f_p(x, Z, U, V, Y) =$

$$= w_1(x, z_1, \ldots, z_p, [u_1, v_1], \ldots, [u_{m-p}, v_{m-p}], y_1, \ldots, y_{k-1}),\ p = 0, 1, \ldots, m.$$

Show that there exists a p such that the polynomial $f_p^*(x, Z, U, V, Y)$ obtained by applying the Razmyslov transform 7.3.16 to $f_p(x, Z, U, V, Y)$ is central for $M_k(K)$.

(ii) Show that the only p with the property as in (i) is $p = 0$.

Hint. (i) Use Exercise 7.3.10 and its proof. See that $f_m(x, Z, U, V, Y)$ is a weak but not a polynomial identity. Show that $f_0([x, t], U, V, Y)$ is a polynomial identity. Therefore $f_p(x, Z, U, V, Y)$ satisfies the assumptions of Lemma of Razmyslov 7.3.17 for some p.

(ii) Very probably there will be difficulties with this exercise. If no success, see the original paper by Halpin [124].

Exercise 7.3.20 Let $\mathrm{char} K = 0$. Find the linearization $w_1(x, z, y)$ of the weak polynomial identity $w(x, y) = [x^2, y]$ for $M_2(K)$ (i.e. the multilinear component of $w(x + z, y)$). Show that $w_1(x, [u, v], y)$ is not a polynomial identity for $M_2(K)$ and $w_1([x, t], [u, v], y)$ is. Apply the Razmyslov transform to $w_1(x, [u, v], y)$ and obtain a central polynomial for $M_2(K)$.

Answer. Up to a multiplicative constant and the names of the variables, one obtains

$$[x_1, x_2][x_3, x_4] + [x_3, x_4][x_1, x_2].$$

Latyshev and Shmelkin [168] constructed a central polynomial in one variable for $M_k(K)$ when the field K is finite.

Exercise 7.3.21 Let $K = \mathbb{F}_q$ be a finite field and let $p(x)$ be an irreducible polynomial of degree k in $\mathbb{F}_q[x]$. Show that the following polynomial in one variable

$$c(x) = \prod_{m=1}^{k-1} (x^{q^m} - x)^{(q^k-1)k} \left(\frac{x^{q^k} - x}{p(x)} \right)^{(q^k-1)k}$$

is central for $M_k(\mathbb{F}_q)$.

Hint. Let $a \in M_k(\mathbb{F}_q)$. Each eigenvalue of a belongs to some extension $\mathbb{F}_{q^m}$ of $\mathbb{F}_q$ with $m \leq k$. Conjugating a by an invertible matrix with entries from an extension of $\mathbb{F}_q$, we consider a in its Jordan normal form. If the eigenvalues of a are not zeros of $p(x)$, then $c(a) = 0$. By Exercises 7.2.3 and 7.2.4, there exists a matrix $a \in M_k(\mathbb{F}_q)$ with eigenvalues among the zeros of $p(x)$. This matrix has no multiple eigenvalues (why?) and hence its Jordan normal form (in $M_k(\mathbb{F}_{q^k})$) is diagonal. Hence $c(a) = g^{q^k-1}(a)$, where $g(a)$ is diagonalizable matrix with nonzero eigenvalues from $\mathbb{F}_{q^k}$. Therefore $c(a) = g^{q^k-1}(a) = e$.

Considering some important algebras, as the generic matrix algebras, a natural question is how big is the centre of the algebra, which gives some rough idea how far from commutative is the algebra. Tracing the literature, one can see that for different purposes one needs different central polynomials.

Problem 7.3.22 Find new central polynomials for the matrix algebras $M_k(K)$, $k \geq 2$.

The complete description of the central polynomials is known for 2×2 matrices only ($\mathrm{char} K = 0$), see e.g. the papers of Formanek [105] and Okhitin [200] for different descriptions.

The next exercises may be useful for searching for central polynomials of low degree and show that over a field of positive characteristic the complete description of the central polynomials should be difficult even for $M_2(K)$.

Exercise 7.3.23 Prove that every central polynomial for $M_k(K)$ is a polynomial identity for $M_{k-1}(K)$.

Hint. Embed the algebra $M_{k-1}(K)$ into $M_k(K)$ assuming that the k-th column and row of $a \in M_{k-1}(K)$ are zero. The evaluation of the central polynomial is a scalar $k \times k$ matrix with 0 in the k-th position of the diagonal.

Exercise 7.3.24 Prove that every central polynomial for $M_k(K)$, $k > 1$, is a polynomial identity for the algebra of $k \times k$ upper triangular matrices $U_k(K)$ with entries from K.

Hint. Let $c(r_1, \ldots, r_n) = \alpha e$, where $0 \neq \alpha \in K$, $r_1, \ldots, r_n \in U_k(K)$. Show that $\alpha e_{11}, \ldots, \alpha e_{k-1,k-1}$ depend on the first $k-1$ rows and columns of $r_1, \ldots, r_n$ only, and do not depend on the last k-th column. Then apply Exercise 7.3.23.

Exercise 7.3.25 (i) Let $f(x_1, x_2, x_3, x_4)$ be a multilinear polynomial of degree 4 in $\mathbb{F}_2\langle X\rangle$. Show that $f(x_1, x_2, x_3, x_4)$ is a central polynomial for $M_2(\mathbb{Z}_2)$ (not necessarily nontrivial) if and only if it is a linear combination of

$$[x_4, x_{\sigma(1)}] \circ [x_{\sigma(2)}, x_{\sigma(3)}],\ \sigma \in S_3, \sigma(2) > \sigma(3),$$

and

$$c(x_1, x_2, x_3, x_4) = [x_4, x_1][x_2, x_3] + [x_4, x_2][x_3, x_1] + [x_4, x_3][x_1, x_2].$$

Show that $c(x_1, x_2, x_3, x_4) = 0$ is a weak but not an ordinary polynomial identity for $M_2(\mathbb{Z}_2)$.

(ii) Show that if $\mathrm{char} K \neq 2$ then the multilinear central polynomials of degree 4 for $M_2(K)$ are linear combinations only of

$$[x_4, x_{\sigma(1)}] \circ [x_{\sigma(2)}, x_{\sigma(3)}],\ \sigma \in S_3, \sigma(2) > \sigma(3).$$

Hint. By Exercise 7.3.24 and by Theorem 5.2.1 for the polynomial identities of the upper triangular matrix algebras (which does not depend on the field if we consider multilinear identities only) a multilinear central polynomial of degree 4 is a linear combination

$$f(x_1,x_2,x_3,x_4)=\sum\alpha_\sigma[x_{\sigma(1)},x_{\sigma(2)}][x_{\sigma(3)},x_{\sigma(4)}],$$

where $\sigma\in S_4$, $\sigma(1)>\sigma(2)$, $\sigma(3)>\sigma(4)$. Replacing x_1,x_2,x_3,x_4 with matrix units and assuming that the result is a scalar matrix, we obtain a homogeneous system with 6 unknowns α_σ. It turns out that the polynomials given above are obtained from solutions of the system and any other solution is their linear combination. We shall illustrate this (with some small additional tricks) in the case of characteristic 2. Since we work with multilinear identities we may assume that $K=\mathbb{Z}_2$. First we write the standard polynomial s_4 in the form

$$s_4(x_1,x_2,x_3,x_4)=\sum[x_4,x_{\sigma(1)}]\circ[x_{\sigma(2)},x_{\sigma(3)}],\ \sigma\in S_3,\sigma(2)>\sigma(3).$$

If $\mathrm{tr}(a_i)=0$, $i=1,2$, by the linearized version of the Cayley-Hamilton theorem (as in the Razmyslov proof of Amitsur-Levitzki Theorem 7.1.6) we obtain that $a_1\circ a_2$ is central, therefore we obtain two more central polynomials of the form

$$[x_4,x_{\sigma(1)}]\circ[x_{\sigma(2)},x_{\sigma(3)}],$$

e.g. for $\sigma(1)=1$ and $\sigma(1)=2$. (The third central polynomial (for $\sigma(1)=3$) is a linear combination of s_4 and the other two). Now, $c(x_1,x_2,x_3,x_4)$ is skew symmetric in x_1,x_2,x_3. We choose a basis of $M_2(\mathbb{Z}_2)$

$$e_{11},e_{12},e_{21},e=[e_{12},e_{21}]$$

We need to calculate only the cases (why? – proper polynomial identities!)

$$c(e_{11},e_{12},e_{21},e_{11})=e,\ c(e_{11},e_{12},e_{21},e_{12})=0.$$

Now we consider the six-dimensional vector space with basis

$$[x_{\sigma(1)},x_{\sigma(2)}][x_{\sigma(3)},x_{\sigma(4)}],\ \sigma\in S_4,\sigma(1)>\sigma(2),\sigma(3)>\sigma(4),$$

and choose a new basis consisting of the four found central polynomials and

$$[x_4,x_1][x_3,x_2],[x_4,x_2][x_3,x_1].$$

It is sufficient to show that every nontrivial linear combination

$$f(x_1,x_2,x_3,x_4)=\alpha[x_4,x_1][x_3,x_2]+\beta[x_4,x_2][x_3,x_1]$$

cannot be a central polynomial. For example

$$f(e_{11},e_{12},e_{11},e_{21})=\alpha e_{22}$$

and $\alpha = 0$. Similarly, $\beta = 0$.

If not explicitly stated, till the end of the chapter we assume that $\mathrm{char} K = 0$.

Problem 7.3.26 Find the minimal degree of the central polynomials for $M_k(K)$, $\mathrm{char} K = 0$. Find the minimal degree of the central polynomials in two variables.

The central polynomials of Formanek [104] are of degree k^2. The original polynomials of Razmyslov [218] were of degree $3k^2 - 1$ but using other weak polynomial identities, Halpin [124] also reduced the degree to k^2. There was a believe that the answer to the above problem is k^2 and this is the case for $k = 1, 2$.

For $k = 3$ Drensky and Azniv Kasparian [91, 92] proved that the minimal degree of the central polynomials is 8. The new central polynomial of degree 8 was obtained using ideas of the Rosset proof of Amitsur-Levitzki Theorem 7.1.8. The approach to show that there are no central polynomials of degree 7 was based on computations as in Exercises 7.3.23 – 7.3.25 combined with techniques of representation theory of the general linear group (see Chapter 12 for the method of representation theory of groups).

Conjecture 7.3.27 (Formanek [108]) The minimal degree of the central polynomials for $M_k(K)$ over a field K of characteristic 0 is equal to

$$\mathrm{mindeg}(M_k(K)) = \frac{1}{2}(k^2 + 3k - 2).$$

Although this is not a reason, the only quadratic function $p(k)$ with $p(1) = 1$, $p(2) = 4$ and $p(3) = 8$ is $\frac{1}{2}(k^2 + 3k - 2)$. What is more important, the conjecture of Formanek agrees with some other conjectures in the theory of PI-algebras. For $k = 4$ Drensky and Giulia Maria Piacentini Cattaneo [94] found a new central polynomial of degree 13. (Pay attention: $\frac{1}{2}(4^2 + 3.4 - 2) = 13$. Unfortunately we do not know whether $M_4(K)$ has central polynomials of degree 12.) Their construction uses a weak polynomial identity of degree 9 and combines the methods of Formanek and Razmyslov. The result was generalized by Drensky [85] to construct central polynomials of degree $(k-1)^2 + 4$ for $M_k(K)$, $k > 2$.

Problem 7.3.28 Find the minimal degree of the weak polynomial identities for $M_k(K)$, $\mathrm{char} K = 0$. Find weak polynomial identities of low degree which induce central polynomials of low degree.

The answer is known for $k = 2$ where the theorem of Razmyslov [217] shows that all weak polynomial identities are consequences from $[x_1^2, x_2] = 0$

(see Remark 7.3.11) and for $k = 3$, where Drensky and Tsetska Rashkova [95] described all weak polynomial identities of degree 6. (There are no weak identities of lower degree for $M_3(K)$). These weak identities give the central polynomials of Halpin (of degree 9) and one of them gives a central polynomial of degree 8.

7.4 Various Identities of Matrices

Exercise 7.4.1 Show that the matrix algebra $M_k(K)$ does not satisfy polynomial identities $f(x, y_1, \ldots, y_{k-1}) = 0$ which are multilinear in $y_1, \ldots, y_{k-1}$. (Remember that the field is of characteristic 0, and hence is infinite.)

Hint. Try to find two different solutions. (i) Replace x by a generic diagonal matrix and the y_p's by the matrix units $e_{p,p+1}$. (ii) Show that f cannot be written as a linear combination of products containing k or more commutators and use that every identity for the matrix algebra is also an identity for the algebra of upper triangular matrices.

Exercise 7.4.2 (Bergman [34]) Let

$$g(u_1, \ldots, u_{k+1}) = \sum \alpha_a u_1^{a_1} \ldots u_{k+1}^{a_{k+1}},\ \alpha_a \in K,$$

be a polynomial in commuting variables and let

$$\theta(g)(x, y_1, \ldots, y_k) = \sum \alpha_a x^{a_1} y_1 x^{a_2} y_2 x^{a_3} y_3 \ldots x^{a_k} y_k x^{a_{k+1}}$$

be the polynomial in $K\langle x, y_1, \ldots, y_k\rangle$, as in the Formanek construction of the central polynomials (Exercise 7.3.4 and Theorem 7.3.5).

(i) Show that $\theta(g)(x, y_1, \ldots, y_k) = 0$ is a polynomial identity for $M_k(K)$ if and only if the polynomial $g(u_1, \ldots, u_{k+1})$ is divisible by all $u_p - u_q$, $1 \leq p < q \leq k + 1$. Show that in this case $\theta(g)(x, y_1, \ldots, y_k)$ follows from the identity of algebraicity in Exercise 2.1.8.

(ii) Let $f(x, y_1, \ldots, y_k) \in K\langle x, y_1, \ldots, y_k\rangle$ be multilinear in $y_1, \ldots, y_k$. Write f in the form

$$f(x, y_1, \ldots, y_k) = \sum \theta(g_i)(x, y_{i_1}, \ldots, y_{i_k}),\ g_i \in K[u_1, \ldots, u_{k+1}].$$

Show that $f(x, y_1, \ldots, y_k) = 0$ is a polynomial identity for $M_k(K)$ if and only if each $\theta(g_i)(x, y_{i_1}, \ldots, y_{i_k}) = 0$ is a polynomial identity.

Hint. (i) Replace x by a diagonal generic matrix and the y_p's with the matrix units

$$e_{12}, e_{23}, \ldots, e_{i-1,i}, \ldots, e_{j-1,j}, e_{ji}, e_{i,j+1}, e_{j+1,j+2}, \ldots, e_{k-1,k},$$

see Fig. 7.2 for the corresponding oriented graph.

Fig. 7.2. The graph corresponding to $\{e_{12}, e_{23}, e_{34}, e_{45}, e_{56}, e_{67}, e_{73}, e_{38}\}$

Show that if $\theta(g) = 0$ is a polynomial identity, then all differences $u_p - u_q$ divide g. On the other hand, if $u_p - u_q$ divides g for all pairs (p, q), $p \neq q$, then replacing x by a diagonal matrix and the y's by matrix units, we obtain 0. Show that the identity of algebraicity corresponds to the product $g = \prod(u_p - u_q)$. Show that

$$\theta(u_p g)(x, y_1, \ldots, y_k) = \theta(g)(x, y_1, \ldots, xy_p, \ldots, y_k) =$$

$$= \theta(g)(x, y_1, \ldots, y_{p-1}x, \ldots, y_k)$$

and derive that all polynomial identities of this form follow from the identity of algebraicity. For (ii) try to modify the above arguments. If some difficulties arise see the original proof of Bergman [34] (or, if his preprint is not available to the reader, see the paper by Drensky and Tsetska Rashkova [95] where a similar result is obtained for weak polynomial identities).

In the next several exercises we again assume that the field K is arbitrary.

Exercise 7.4.3 Show that $K\langle x, y\rangle$ has a basis

$$\{x^{p_0}y^{q_0}[x, y]x^{p_1}y^{q_1}[x, y] \ldots x^{p_{k-1}}y^{q_{k-1}}[x, y]x^{p_k}y^{q_k} \mid p_i, q_i \geq 0, k \geq 0\}.$$

Hint. Using the equality $yx = xy - [x, y]$, we can write any element of $K\langle x, y\rangle$ as a linear combination of the above elements. In order to see that the elements are linearly independent, use one of the following two possibilities. (i) Show that any nontrivial linear combination of these elements does not vanish on some algebra of generic upper triangular matrices (as in Theorem

5.2.1). (ii) Assume for a while that these elements are linearly independent. Compute the Hilbert series of the vector space spanned by them. It is

$$H(t)=\frac{1}{(1-t)^2}\sum_{k\geq 0}\left(\frac{t^2}{(1-t)^2}\right)^k=$$

$$=\frac{1}{1-2t}=\mathrm{Hilb}(K\langle x,y\rangle,t).$$

If our elements are linearly dependent, this will give us that some of the coefficients of the Hilbert series of the vector space spanned by them are smaller than the corresponding coefficients of the Hilbert series of $K\langle x,y\rangle$.

Exercise 7.4.4 Let

$$R=K\langle x,y\rangle/([x^2,y],[y^2,x])$$

be the factor algebra of $K\langle x,y\rangle$ modulo the ideal generated by $[x^2,y]$ and $[y^2,x]$. Show that as a vector space R has a basis

$$\{x^ay^b[x,y]^c\mid a,b,c\geq 0\}.$$

Hint. We work in R using the same symbols x and y for the images of the free generators of $K\langle x,y\rangle$. Since

$$0=[x^2,y]=x[x,y]+[x,y]x,\ 0=[y^2,x]=-y[x,y]-[x,y]y,$$

we obtain that

$$[x,y]x=-x[x,y],\ [x,y]y=-y[x,y].$$

Applying Exercise 7.4.3, we see that R is spanned on

$$\{x^ay^b[x,y]^c\mid a,b,c\geq 0\}.$$

First, let $\mathrm{char}K\neq 2$. The algebra generated by two "generic" traceless matrices of the form

$$u=\begin{pmatrix}\xi & 0\\ 0 & -\xi\end{pmatrix},\ v=\begin{pmatrix}\eta & \zeta\\ \zeta & -\eta\end{pmatrix},$$

(where ξ,η,ζ are algebraically independent commuting variables) satisfies the defining relations of the algebra R (check it!). Assuming that $\xi>\eta>\zeta$, show that the leading term of $u^av^b[u,v]^c$ is equal to

$$\alpha\xi^{a+c}\eta^b\zeta^c(e_{11}-e_{22})^{a+b}[e_{11}-e_{22},e_{12}+e_{21}]^c,\ 0\neq\alpha\in K.$$

Derive from here that the elements $x^ay^b[x,y]^c$ are linearly independent in R. If $\mathrm{char}K=2$, use the generic traceless matrices

$$u_1=\xi_1(e_{11}+e_{22})+\eta_1(e_{12}+e_{21}),u_2=\xi_2(e_{11}+e_{22})+\eta_2e_{12}$$

and the ordering $\xi_1 > \eta_1 > \xi_2 > \eta_2$.

Remark 7.4.5 Exercise 7.4.4 shows that the factor algebra of $K\langle x, y\rangle$ modulo the ideal generated by $[x^2, y]$ and $[y^2, x]$ is isomorphic to the factor algebra of $K\langle x, y\rangle$ modulo the weak polynomial identities of $M_2(K)$. This is a partial case of a general result concerning weak polynomial identities of 2×2 matrices, see Exercise 12.6.12 and Remark 12.6.13.

Exercise 7.4.6 Show that the following two systems of elements in $K\langle x, y\rangle$ are independent (i.e. are systems of free generators in subalgebras of $K\langle x, y\rangle$).

$$y, yx, yx^2, yx^3, \ldots,$$

$$y^k[x, y]x^l,\ k, l \geq 0.$$

Hint. Show that the algebra generated by $y, yx, yx^2, yx^3, \ldots$ is spanned by the words $yx^{i_1}\ldots yx^{i_p}$ and the presentation

$$yx^{i_1}\ldots yx^{i_p} = (yx^{i_1})\ldots(yx^{i_p})$$

is unique. For the second set use similar approach applying Exercise 7.4.3.

Exercise 7.4.7 Show that every PI-algebra satisfies a polynomial identity in two variables.

Hint. If R satisfies no polynomial identity in two variables, then

$$K\langle x, y\rangle \cap T(R) = 0,\ F_2(R) = K\langle x, y\rangle/(K\langle x, y\rangle \cap T(R)) \cong K\langle x, y\rangle$$

and by Exercise 7.4.6, $F_2(R)$ contains a free associative subalgebra. Hence both $F_2(R)$ and R satisfy no polynomial identities at all.

Exercise 7.4.8. Give a new proof of Exercise 7.1.13. Show that the polynomials

$$s_{2k}([x, y], y[x, y], [x, y]x, y^2[x, y], y[x, y]x, [x, y]x^2, y^3[x, y], y^2[x, y]x, \ldots)$$

are nonzero in $K\langle x, y\rangle$ and, for k sufficiently large, are of degree less than the degree of the identity of algebraicity for $M_k(K)$.

Hint. Use Exercise 7.4.6 and the fact that

$$\deg s_{2k}([x, y], y[x, y], [x, y]x, \ldots) = \mathcal{O}(k\sqrt{k})$$

and the degree of the identity of algebraicity is $\mathcal{O}(k^2)$.

We complete this section with short discussions on other T-ideals in some sense similar to the matrix T-ideals.

Remark 7.4.9 Over a field of characteristic 0, Latyshev [166] called a T-ideal $T(R)$ *stable* if the set of its multilinear polynomials is stable under the Razmyslov transform 7.3.16. He started the systematic study of stable T-ideals. Further development was given by Okhitin [199], who proved that the stable T-ideals have many properties similar to those of the T-ideals of polynomial identities of matrices. In particular, he established that if $T(R)$ is stable, then R has central polynomials.

8. Multilinear Polynomial Identities

The starting point of our considerations was the claim that the class of PI-algebras is reasonably big and enjoys the most important properties of commutative and of finite dimensional algebras. One of the goals of this and the next chapters is to give a quantitative confirmation of this statement.

In this chapter we consider multilinear polynomial identities. Over a field of positive characteristic not all polynomial identities of an algebra follow from the multilinear ones. Nevertheless we shall see that the multilinear identities carry a lot of information about all identities. Up till now, most of our considerations have involved computing with the identities of concrete important algebras. Now we go from one extreme to the other and assume that the only information we have is that the algebra satisfies *some* polynomial identity. The main quantitative result of the chapter is the theorem of Regev for the exponential growth of the codimension sequence of a PI-algebra. As a consequence we give another important theorem of Regev that the tensor product of two PI-algebras is again PI. This is one more confirmation that the class of PI-algebras is nice: it is closed with respect to natural algebraic operations. Then, till the end of the section we consider algebras over a field of characteristic 0. First we study the PI-algebras with polynomial growth of the codimension sequence and give some description in the unitary case. Then we prove one of the corner stones of the PI-theory and its applications: the Nagata-Higman theorem that a nonunitary algebra which is nil of bounded index, is nilpotent. Finally, we give some (very slight) idea about the structure theory of T-ideals developed by Kemer and prove his theorem that the standard identity implies the Capelli identity.

8.1 The Codimension Theorem of Regev

In this section we assume that the base field K is arbitrary. Although we still have our convention to consider unitary algebras, the main results hold without any changes also in the nonunitary case. First we prove the theorem of Regev [222] for the exponential growth of the codimension sequence of a PI-algebra, i.e. for every PI-algebra R there exists an $a > 0$ such that $c_n(R) = \mathcal{O}(a^n)$. (Compare the asymptotic behaviour of $c_n(R)$ with this of

$\dim P_n = n!$ which grows much faster than any exponential function.) We give the proof of Latyshev [163] based on the Dilworth theorem in combinatorics [61]. The version of the Dilworth theorem used in our exposition is proposed by Amitsur. This approach to the proof of the theorem of Regev is considered to be standard nowadays.

Definition 8.1.1 A set P with a relation $\prec$ (notation $(P, \prec)$) is *partially ordered* if the relation is transitive, i.e. $a \prec b$ and $b \prec c$ for $a, b, c \in P$ implies $a \prec c$ and if the relations $a \prec b$ and $b \prec a$ cannot be satisfied simultaneously for any $a, b \in P$. The elements $a_1, \ldots, a_k$ form a *chain* in P if $a_1 \prec \ldots \prec a_k$; they form an *antichain* if the relation $a_i \prec a_j$ does not hold for any pair (a_i, a_j) with different i and j.

Below we give without proof the original version of the Dilworth theorem.

Theorem 8.1.2 (Dilworth [61]) *Let $(P, \prec)$ be a partially ordered set. Then the number of chains of P into which P can be partitioned (i.e. presented as a disjoint union) is equal to the maximum number of elements in an antichain of P.*

One idea for the following definition comes from the Shirshov approach to combinatorics of words and its applications to PI-algebras. (Compare Definition 8.1.3 with Definition 9.1.5.)

Definition 8.1.3 For a permutation π in S_n we denote by $d(\pi)$ the largest number d for which there exist integers $1 \leq i_1 < i_2 < \ldots < i_d \leq n$ such that $\pi(i_1) > \pi(i_2) > \ldots > \pi(i_d)$. For a fixed integer $d > 1$, the permutation π in S_n is called *d-good* if $d(\pi) < d$. In other words, for a fixed $\pi \in S_n$, we introduce a partial ordering on the set $P = \{1, 2, \ldots, n\}$ in the following way: $i \prec j$ if and only if $i < j$ and $\pi(i) < \pi(j)$. Then $d(\pi)$ is the maximum length of the antichains in $\{1, 2, \ldots, n\}$ and π is d-good if there are no antichains of length d.

For example, let $n = 6$ and

$$\pi = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 5 & 3 & 2 & 4 & 1 & 6 \end{pmatrix}.$$

Then $d(\pi) = 4$, since $1 < 2 < 3 < 5$ and $\pi(1) > \pi(2) > \pi(3) > \pi(5)$ (and there are no antichains of length 5). Hence π is 5-good (and also 6-good).

Definition 8.1.4 For a permutation $\pi \in S_n$ we construct a pair of tables $T_1(\pi) = (t_{ij})$ and $T_2(\pi) = (u_{ij})$ in the following way. We set $t_{11} = 1$, $u_{11} = \pi(1)$. By induction, if t_{1j} exists, it is the smallest k such that $t_{1,j-1} < k \leq n$ and $\pi(k) > u_{1,j-1}$. Then we set $u_{1j} = \pi(k)$. When we cannot find the next t_{1j} we start with the second rows of $T_1(\pi)$ and $T_2(\pi)$: t_{21} is the smallest

$k \in \{1, 2, \ldots, n\}$ which does not belong to the first row of $T_1(\pi)$, $u_{21} = \pi(t_{21})$ and, if $j > 1$, then t_{2j} is the smallest k not in the first row of $T_1(\pi)$ such that $t_{2,j-1} < k \leq n$ and $\pi(k) > u_{2,j-1}$; we set $u_{2j} = \pi(k)$. When we finish the second row, we continue with the third, etc.

For example, let $n = 6$ and

$$\pi = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 4 & 3 & 5 & 1 & 2 & 6 \end{pmatrix}.$$

We give the consecutive steps for constructing $T_1(\pi)$ and $T_2(\pi)$:

$t_{11} = 1$, $u_{11} = \pi(t_{11}) = 4$:

1. $$T_1 = (1), T_2 = (4);$$

$\pi(2) = 3 < 4 = u_{11}$, $\pi(3) = 5 > 4 = u_{11}$, hence $t_{12} = 3$, $u_{12} = 5$:

2. $$T_1 = (1\,3), T_2 = (4\,5);$$

$\pi(4) = 1 < 5 = u_{12}$, $\pi(5) = 2 < 5 = u_{12}$, $\pi(6) = 6 > 5 = u_{12}$, hence $t_{13} = 6$, $u_{13} = \pi(6) = 6$:

3. $$T_1 = (1\,3\,6), T_2 = (4\,5\,6).$$

We cannot continue the process. The integers left in π are

$$\pi = \begin{pmatrix} * & 2 & * & 4 & 5 & * \\ * & 3 & * & 1 & 2 & * \end{pmatrix}$$

and we start with the second rows of $T_1(\pi)$ and $T_2(\pi)$:
$t_{21} = 2$, $u_{21} = \pi(t_{21}) = 3$:

4. $$T_1 = \begin{pmatrix} 1 & 3 & 6 \\ 2 & & \end{pmatrix}, T_2 = \begin{pmatrix} 4 & 5 & 6 \\ 3 & & \end{pmatrix}.$$

Since $\pi(4) = 1 < 3 = u_{21}$, $\pi(5) = 2 < 3 = u_{21}$, the process stops again and, considering

$$\pi = \begin{pmatrix} * & * & * & 4 & 5 & * \\ * & * & * & 1 & 2 & * \end{pmatrix},$$

we continue with the third rows. We set $t_{31} = 4$, $u_{31} = \pi(t_{31}) = 1$:

5. $$T_1 = \begin{pmatrix} 1 & 3 & 6 \\ 2 & & \\ 4 & & \end{pmatrix}, T_2 = \begin{pmatrix} 4 & 5 & 6 \\ 3 & & \\ 1 & & \end{pmatrix}.$$

Since $\pi(5) = 2 > 1 = u_{31}$ we complete the third rows of the tables $T_1(\pi)$ and $T_2(\pi)$ by $t_{32} = 5$, $u_{32} = \pi(t_{32}) = 2$:

6. $$T_1 = \begin{pmatrix} 1 & 3 & 6 \\ 2 & & \\ 4 & 5 & \end{pmatrix}, T_2 = \begin{pmatrix} 4 & 5 & 6 \\ 3 & & \\ 1 & 2 & \end{pmatrix}.$$

Lemma 8.1.5 (Amitsur, see [231]) *For a permutation $\pi \in S_n$, the integer $d(\pi)$ is equal to the number of the rows d of the table $T_1(\pi)$.*

Proof. By the construction of the tables $T_1(\pi)$ and $T_2(\pi)$, if $i_1 < \ldots < i_{d(\pi)}$ and $\pi(i_1) > \ldots > \pi(i_{d(\pi)})$, then $i_1, \ldots, i_{d(\pi)}$ are in different rows of $T_1(\pi)$. Hence $d(\pi) \leq d$. Now we shall construct a sequence $i_1 < \ldots < i_d$ with $\pi(i_1) > \ldots > \pi(i_d)$. We start with $i_d = t_{d1}$ and, by induction, if i_{k+1} is in the $k+1$-st row of $T_1(\pi)$, we define $i_k = t_{kj}$ for the largest j such that $t_{kj} < i_{k+1}$. If $u_{kj} < \pi(i_{k+1})$, then i_{k+1} should be in the k-th row of $T_1(\pi)$, which is not true. Therefore,

$$\pi(i_k) = \pi(t_{kj}) = u_{kj} > \pi(i_{k+1})$$

and we can continue the process. This gives that $d(\pi) = d$.

Theorem 8.1.6 (Latyshev [163]) *If R is a PI-algebra and the T-ideal $T(R)$ contains a polynomial identity of degree d, then the vector space of multilinear polynomials of degree n in $K\langle X\rangle$ is spanned modulo $T(R)$ by the monomials $x_{\pi(1)} \ldots x_{\pi(n)}$ where $\pi \in S_n$ is d-good (i.e. $d(\pi) < d$).*

Proof. Since $T(R)$ contains a polynomial identity of degree d, it contains also a multilinear polynomial of degree d. Without loss of generality we may assume (why?) that R satisfies an identity of the form

$$x_d x_{d-1} \ldots x_1 = \sum_{\sigma \in S_d} \alpha_\sigma x_{\sigma(1)} x_{\sigma(2)} \ldots x_{\sigma(d)},\ \alpha_\sigma \in K,$$

and the summation is over all permutations σ different from $\delta \in S_d$, where

$$\delta = \begin{pmatrix} 1 & 2 & \ldots & d-1 & d \\ d & d-1 & \ldots & 2 & 1 \end{pmatrix}.$$

We shall work in $P_n(R) = P_n/(P_n \cap T(R))$, i.e. in the vector space of multilinear polynomials of degree n modulo the polynomial identities of R. Let G_d be the set of all monomials $x_{\pi(1)} \ldots x_{\pi(n)} \in P_n(R)$ such that the permutation $\pi \in S_n$ is d-good. We shall show that $P_n(R) = \text{span}(G_d)$. We order lexicographically the monomials in $P_n(R)$ assuming that

$$x_{\sigma(1)} \ldots x_{\sigma(n)} < x_{\tau(1)} \ldots x_{\tau(n)}$$

if and only if for some k

$$\sigma(1) = \tau(1), \ldots, \sigma(k) = \tau(k), \sigma(k+1) < \tau(k+1).$$

Let $h = x_{\pi(1)} \dots x_{\pi(n)}$ be the minimal monomial which is not in $\mathrm{span}(G_d)$. Hence $d(\pi) \geq d$ and there exist $i_1 < \dots < i_d$ with $\pi(i_1) > \dots > \pi(i_d)$. We write h in the form

$$h = h_0 x_{\pi(i_1)} h_1 x_{\pi(i_2)} h_2 \dots h_{d-1} x_{\pi(i_d)} h_d,$$

apply the polynomial identity of degree d for

$$\bar{x}_d = x_{\pi(i_1)} h_1, \bar{x}_{d-1} = x_{\pi(i_2)} h_2, \dots, \bar{x}_2 = x_{\pi(i_{d-1})} h_{d-1}, \bar{x}_1 = x_{\pi(i_d)} h_d$$

and obtain

$$h = h_0 \cdot (\bar{x}_d \bar{x}_{d-1} \dots \bar{x}_1) = \sum_{\sigma \in S_d} \alpha_\sigma h_0 \bar{x}_{\sigma(1)} \bar{x}_{\sigma(2)} \dots \bar{x}_{\sigma(d)} =$$

$$= \sum_{\sigma \in S_d} \alpha_\sigma h_0 x_{\pi(i_{\sigma(d)})} h_{\sigma(d)} x_{\pi(i_{\sigma(d-1)})} h_{\sigma(d-1)} \dots x_{\pi(i_{\sigma(1)})} h_{\sigma(1)}.$$

Since $\pi(i_1) > \dots > \pi(i_d)$ and the summation is on $\sigma \neq \delta$, we obtain that all monomials in the latter sum are below than h in the lexicographic ordering and, by inductive arguments, belong to the vector subspace $\mathrm{span}(G_d)$ spanned by the set G_d corresponding to the d-good permutations. Hence h also belongs to $\mathrm{span}(G_d)$. This completes the proof of the theorem.

Now we are ready to prove the theorem of Regev for the exponential growth of the codimensions of a PI-algebra R with the estimate obtained by Latyshev.

Theorem 8.1.7 (Regev [222]) *Let R be a PI-algebra with a polynomial identity of degree d. Then the sequence of the (multilinear) codimensions of the polynomial identities of R satisfies*

$$c_n(R) \leq (d-1)^{2n}, \; n = 0, 1, 2, \dots$$

Proof. By Theorem 8.1.6, it is sufficient to show that the number of the d-good permutations in S_n is bounded by $(d-1)^{2n}$. By Lemma 8.1.5, for any d-good permutation π, the tables $T_1(\pi) = (t_{ij})$ and $T_2(\pi) = (u_{ij})$ constructed in Definition 8.1.4, have less than d rows. Since $\pi(t_{ij}) = u_{ij}$, every permutation π is uniquely determined by the pair $(T_1(\pi), T_2(\pi))$. In each row of $T_1(\pi)$ and $T_2(\pi)$ the integers $t_{i1}, t_{i2}, \dots$ and $u_{i1}, u_{i2}, \dots$ increase. Each integer $1, 2, \dots, n$ can be written in one of the $d-1$ rows of $T_1(\pi)$ and in one of the $d-1$ rows of $T_2(\pi)$ (maybe not all pairs of tables correspond to substitutions). Hence the number of the pairs of tables with less than d rows is bounded by $(d-1)^{2n}$. This completes the proof of the theorem.

Exercise 8.1.8 Prove directly, using Theorem of Dilworth 8.1.2, that the number of the d-good permutations in S_n is bounded by $(d-1)^{2n}$.

Hint. Let π be d-good. Consider the partial ordering in $\{1,2,\ldots,n\}$ induced by π. Then the maximum length of the antichains in π is $\leq d-1$ and by the Dilworth theorem, the set $\{1,2,\ldots,n\}$ can be partitioned in $\leq d-1$ chains, say $C_1,\ldots,C_p$, $p\leq d-1$. Every chain corresponds to a sequence of pairs $(i_1,\pi(i_1)),\ldots,(i_k,\pi(i_k))$ with $i_1<\ldots<i_k$ and $\pi(i_1)<\ldots<\pi(i_k)$. Define functions $t:\{1,2,\ldots,n\}\to\{1,\ldots,d-1\}$ and $u:\{1,2,\ldots,n\}\to\{1,\ldots,d-1\}$ by the rule $t(i)=j$ and $u(\pi(i))=j$ if $(i,\pi(i))$ is in the j-th chain C_j. Show that different d-good permutations correspond to different pairs of functions (t,u) and the total number of pairs of functions is $(d-1)^{2n}$.

Exercise 8.1.9 Show that for any PI-algebra R, the codimension series of R

$$c(R,t)=\sum_{n\geq 0}c_n(R)t^n$$

has a positive radius of convergence.

Hint. Use that if R satisfies a polynomial identity of degree d, then the codimension series is majored by

$$\sum_{n\geq 0}(d-1)^{2n}t^n=\frac{1}{1-(d-1)^2t}.$$

It is remarkable, that using purely quantitative approach, Regev [222] succeeded to prove a purely qualitative result and to answer in the affirmative the problem of Kaplansky whether the tensor product of two PI-algebras is again PI.

Theorem 8.1.10 (Regev [222]) *The tensor product $R=R_1\otimes_K R_2$ of two PI-algebras R_1 and R_2 is also a PI-algebra.*

Proof. Let R_1 and R_2 satisfy, respectively, polynomial identities of degree d_1 and d_2. Therefore, by Theorem of Regev 8.1.7,

$$c_n(R_1)\leq(d_1-1)^{2n},\ c_n(R_2)\leq(d_2-1)^{2n}.$$

We choose n such that $c_n(R_1)c_n(R_2)<n!$. This is always possible because a^n is smaller than $n!$ for large n. Let

$$\{g_i(x_1,\ldots,x_n)\mid i=1,2,\ldots,c'=c_n(R_1)\},$$

$$\{h_j(x_1,\ldots,x_n)\mid j=1,2,\ldots,c''=c_n(R_2)\}$$

be bases respectively of $P_n(R_1)$ and $P_n(R_2)$. For every permutation $\pi\in S_n$, we consider $x_{\pi(1)}\ldots x_{\pi(n)}$ as an element of $P_n(R_1)$ and write it as a linear combination

$$x_{\pi(1)}\dots x_{\pi(n)}=\sum_{i=1}^{c'}\beta_{\pi i}g_i(x_1,\dots,x_n),\ \beta_{\pi i}\in K.$$

Similarly, in $P_n(R_2)$,

$$x_{\pi(1)}\dots x_{\pi(n)}=\sum_{j=1}^{c''}\gamma_{\pi j}h_j(x_1,\dots,x_n),\ \gamma_{\pi j}\in K.$$

Pay attention, that the above two equations are polynomial identities respectively for R_1 and R_2, i.e. they are automatically satisfied for any $u_1,\dots,u_n\in R_1$ and $v_1,\dots,v_n\in R_2$, respectively. We look for a multilinear polynomial identity of degree n for the tensor product $R=R_1\otimes R_2$ of the K-algebras R_1 and R_2. Let

$$f(x_1,\dots,x_n)=\sum_{\pi\in S_n}\xi_\pi x_{\pi(1)}\dots x_{\pi(n)}=0$$

be the desired polynomial identity for R, where the ξ_π's are unknown coefficients from K. Since $f=0$ is multilinear, it is sufficient to show that it vanishes for arbitrary

$$u_1\otimes v_1,u_2\otimes v_2,\dots,u_n\otimes v_n,\ u_1,\dots,u_n\in R_1,\ v_1,\dots,v_n\in R_2.$$

We calculate $f(u_1\otimes v_1,\dots,u_n\otimes v_n)$ and obtain

$$f(u_1\otimes v_1,\dots,u_n\otimes v_n)=\sum_{\pi\in S_n}\xi_\pi(u_{\pi(1)}\otimes v_{\pi(1)})\dots(u_{\pi(n)}\otimes v_{\pi(n)})=$$

$$=\sum_{\pi\in S_n}\xi_\pi(u_{\pi(1)}\dots u_{\pi(n)})\otimes(v_{\pi(1)}\dots v_{\pi(n)})=$$

$$=\sum_{\pi\in S_n}\sum_{i=1}^{c'}\sum_{j=1}^{c''}\xi_\pi\beta_{\pi i}\gamma_{\pi j}g_i(u_1,\dots,u_n)h_j(v_1,\dots,v_n)=0.$$

We rewrite this equation in the form

$$\sum_{i=1}^{c'}\sum_{j=1}^{c''}\left(\sum_{\pi\in S_n}\beta_{\pi i}\gamma_{\pi j}\xi_\pi\right)g_i(u_1,\dots,u_n)h_j(v_1,\dots,v_n)=0.$$

Let us consider the linear homogeneous system

$$\sum_{\pi\in S_n}\beta_{\pi i}\gamma_{\pi j}\xi_\pi=0,\ i=1,\dots,c',j=1,\dots,c''.$$

The number of the unknowns ξ_π is $n!$ and the number of the equations is

$$c'c''=c_n(R_1)c_n(R_2)<n!.$$

Hence the system has a nonzero solution ξ_π, $\pi \in S_n$. Clearly, any nontrivial solution of the system corresponds to a nontrivial polynomial identity for $R = R_1 \otimes R_2$.

Additional information about the polynomial identities of the tensor product of two PI-algebras over a field of characteristic 0 can be found in the paper of Regev [225]. For generalizations in positive characteristic see the paper of Di Vincenzo [62].

8.2 Algebras with Polynomial Growth of Codimensions

Till the end of the chapter, if not explicitly stated, we assume that the field K is of characteristic 0.

Kemer [138] described the T-ideals with polynomial growth of the codimension sequence, i.e. the T-ideals $T(R)$ such that $c_n(R) = \mathcal{O}(n^k)$ for some k. The result is given in the language of representation theory of the symmetric group discussed in Chapter 12. Another description also due to Kemer [139] is given below. In particular, one can see from there that the growth of the codimensions of a PI-algebra is either polynomial or exponential.

Theorem 8.2.1 (Kemer [139]) *A PI-algebra R has a polynomial growth of the codimension sequence if and only if R satisfies polynomial identities $f_1 = 0$ and $f_2 = 0$ such that $f_1 = 0$ is not an identity for the Grassmann algebra and $f_2 = 0$ is not an identity for the algebra of 2×2 upper triangular matrices.*

For unitary algebras the description of the T-ideals with polynomial growth of the proper codimensions $\gamma_n(R)$ was given by Nadzharyan [187]. For Lie algebras the T-ideals with polynomial growth of the codimensions were characterized by Benediktovich and Zalesskiy [28].

Exercise 8.2.2 Show that if a PI-algebra R has a polynomial growth of the codimension sequence, then R satisfies polynomial identities $f_1 = 0$ and $f_2 = 0$ such that $f_1 = 0$ is not an identity for the Grassmann algebra and $f_2 = 0$ is not an identity for the algebra of 2×2 upper triangular matrices. If, additionally, R is unitary, then R satisfies the standard identity $s_{2k} = 0$ and the Engel identity $[y, x, \ldots, x] = y\mathrm{ad}^l(x) = 0$ for some k and l.

Hint. Use the results of Chapter 5. If all polynomial identities of R hold for the Grassmann algebra E, then $c_n(R) \geq c_n(E) = 2^{n-1}$, $n = 1, 2, \ldots$; similarly for the algebra of upper triangular matrices. Hence the growth of the codimension sequence of the identities of R would be exponential.

Now, let R be unitary. Then R satisfies a proper multilinear polynomial identity which does not hold for the 2×2 upper triangular matrices. This identity is of the form

$$f(x_1,\ldots,x_n)=\sum_{i=2}^{n}\alpha_i[x_i,x_1,x_2,\ldots,\hat{x}_i,\ldots,x_n]+\ldots,\ \alpha_i\in K,$$

where $\hat{x}_i$ means that x_i is missing, some of the α_i's are different from 0 and $\ldots$ is for the summands which are products of more than one commutator. Replace $x_i=y$ and the other x_j's by x. Similarly, if a proper multilinear polynomial identity does not hold for the Grassmann algebra, it is equivalent to

$$f(x_1,\ldots,x_{2n})=[x_1,x_2]\ldots[x_{2n-1},x_{2n}]+$$
$$+\sum\alpha_{i,\sigma}x_{\sigma(1)}\ldots x_{\sigma(i_0)}\times$$
$$\times[x_{\sigma(i_0+1)}\ldots x_{\sigma(i_1)},x_{\sigma(i_1+1)}\ldots x_{\sigma(i_2)},x_{\sigma(i_2+1)}\ldots x_{\sigma(i_3)}]x_{\sigma(i_3+1)}\ldots x_{\sigma(2n)}.$$

Show that

$$\sum_{\pi\in S_k}(\mathrm{sign}\pi)[x_{\pi(1)}\ldots x_{\pi(j_1)},x_{\pi(j_1+1)}\ldots x_{\pi(j_2)},x_{\pi(j_2+1)}\ldots x_{\pi(k)}]=0.$$

Exercise 8.2.3 Let R be a unitary algebra such that $T(R)$ contains Γ_{2k} for some $k\geq 1$ (where Γ_{2k} is the vector space of all proper multilinear polynomials of degree $2k$ in $K\langle X\rangle$). Show that $T(R)$ contains Γ_n for all $n\geq 2k$.

Hint. Let $u=[x_{\sigma(1)},\ldots,x_{\sigma(p)}]\ldots[\ldots,x_{\sigma(n)}]$, $\sigma\in S_n$, $n>2k$, be a product of commutators. If some of the commutators is of length larger than 2, e.g. the first commutator is of length $p>2$, then it is a consequence of

$$v=[x_1,\ldots,x_{p-1}]\ldots[\ldots,x_{n-1}]\in\Gamma_{n-1}\subset T(R),$$

where the length of the first commutator of v is $p-1$ and the other commutators are of the same length as in u. If all commutators of u are of length 2, then u follows from

$$[x_1,x_2]\ldots[x_{2k-1},x_{2k}]\in\Gamma_{2k}\subset T(R).$$

Exercise 8.2.4 Show that if the codimension sequence of the unitary PI-algebra R is not of polynomial growth, then

$$c_n(R)\geq 2^{n-1},\ n=1,2,\ldots$$

Hint. If $\gamma_{2k}(R)=0$ for some k, then $\gamma_p(R)=0$ for all $p>2k$ and, by Theorem 4.3.12 (ii), $c_n(R)$ is of polynomial growth. If $\gamma_{2k}(R)\neq 0$ for all k, then $\gamma_n(R)\geq\gamma_n(E)$ and, hence $c_n(R)\geq c_n(E)=2^{n-1}$.

Exercise 8.2.5 (Drensky [80]) Show that, if the codimension sequence of the unitary algebra R is polynomially bounded, then for some $k\in\mathbb{N}\cup\{0\}$,

$$c_n(R) = \alpha_k n^k + \mathcal{O}(n^{k-1}),\ 0 \neq \alpha_k \in \mathbb{Q}.$$

Hint. If $c_n(R)$ is polynomially bounded then, by Exercise 8.2.4, $\gamma_p(R) = 0$ for p sufficiently large. Let k be the greatest integer such that $\gamma_k(R) \neq 0$. By Theorem 4.3.12 (ii), $c_n(R)$ is a polynomial of degree k in $\mathbb{Q}[n]$.

Exercise 8.2.6 (Specht [245]) Prove that the dimension of the vector space Γ_n of the proper multilinear polynomials of degree n in the free associative algebra is

$$\gamma_n = \dim \Gamma_n = n! \left(1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \ldots + (-1)^n \frac{1}{n!}\right).$$

Hint. Apply Theorem 4.3.12 (ii). Use the formula $\tilde{c}(t) = e^t \tilde{\gamma}(t)$ for the exponential generating functions (which holds not only for relatively free, but also for free associative algebras), where $c_n = n!$ and $\tilde{c}(t) = 1 + t + t^2 + \ldots$ Hence

$$\tilde{\gamma}(t) = e^{-t}(1 + t + t^2 + t^3 + \ldots).$$

Decompose e^{-t} into a series and multiply it by $1 + t + t^2 + \ldots$.

Exercise 8.2.7 (Drensky and Regev [96]) Show that in Exercise 8.2.5

$$\alpha_k \leq 1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \ldots + (-1)^k \frac{1}{k!},$$

which, for k sufficiently large, is close to e^{-1}.

Hint. Apply Exercises 8.2.5 and 8.2.6 and use that $\alpha_k \leq \gamma_k$.

The next exercise gives a construction which is a special case of the *wreath product* of Lie algebras introduced by Shmelkin [240].

Exercise 8.2.8 Let $U = U(F(\mathfrak{N}_2))$ be the universal enveloping algebra of the free nilpotent of class two Lie algebra $F = F(\mathfrak{N}_2)$ of infinite rank. Let M be the free right U-module generated by one element a. Make the direct sum $G = M \oplus F$ a Lie algebra assuming that

$$[au_1 + f_1, au_2 + f_2] = a(u_1 f_2 - u_2 f_1) + [f_1, f_2],\ u_1, u_2 \in U, f_1, f_2 \in F.$$

Show that G belongs to the variety $\mathfrak{A}\mathfrak{N}_2$ defined by the polynomial identity

$$[[x_1, x_2, x_3], [x_4, x_5, x_6]] = 0.$$

Hint. Recall that F is spanned by x_i and $[x_j, x_k]$ and satisfies the relations $[x_i, x_j, x_k] = 0$. Check the defining identity of $\mathfrak{A}\mathfrak{N}_2$ on the basis elements of G.

The following exercise presents a modification of an example of Volichenko [263] which shows one more difference between the behaviour of the codimensions of associative and Lie algebras.

Exercise 8.2.9 Show that the codimension sequence of the variety of Lie algebras $\mathfrak{A}\mathfrak{N}_2$ grows faster than any exponential function.

Hint. Consider the polynomials in the free Lie algebra

$$[[x_{2n+1}, x_{2n+2}, x_{2n+3}], [x_{n+1}, x_{\sigma(1)}], [x_{n+2}, x_{\sigma(2)}], \ldots, [x_{2n}, x_{\sigma(n)}]], \ \sigma \in S_n.$$

Find linearly independent images of these polynomials in the algebra G from Exercise 8.2.8. The number of the polynomials is $n!$ and $c_{2n+3}(\mathfrak{A}\mathfrak{N}_2) \geq n!$.

Remark 8.2.10 The variety of Lie algebras $\mathfrak{A}\mathfrak{N}_2$ has another remarkable property discovered also by Volichenko [264]. Its codimension sequence grows faster than any exponential function but any of its proper subvarieties has an exponential growth of the codimension sequence ($\mathrm{char} K = 0$).

Explicit formulas for the codimensions of a given algebra R are known in few cases only. As we have seen, a lot of information about the polynomial identities of an algebra can be obtained from the asymptotic behaviour of the codimension sequence. The main results here are due to Regev who did a lot of work in this direction. In particular, Regev determined the exact asymptotics of the codimensions of the $k \times k$ matrix algebras over a field of characteristic 0 (see his survey article [226]). The approach of Regev is based on representation theory of the symmetric group, invariant theory of matrices, combinatorics and analytic methods, for example evaluating of multiple integrals.

As a first approximation of the asymptotic behaviour of a given sequence of positive numbers $a_0, a_1, a_2, \ldots$ one may consider the radius of convergence $r(a)$ of the generating function $a(t) = \sum_{n\geq 0} a_n t^n$. Since

$$r(a) = \frac{1}{\limsup_{n\to\infty} \sqrt[n]{a_n}},$$

we can measure the growth of $a_0, a_1, a_2, \ldots$ by $\limsup_{n\to\infty} \sqrt[n]{a_n}$. There is a conjecture that $\limsup_{n\to\infty} \sqrt[n]{c_n(R)}$ is integer for any PI-algebra R. This is known for many important PI-algebras. Recently Giambruno and Zaicev [116] have established that $\limsup_{n\to\infty} \sqrt[n]{c_n(R)}$ is integer for any finitely generated PI-algebra R. Their proof uses the theory of Kemer (see Section 8.4) combined with the asymptotic methods of Regev.

8.3 The Nagata-Higman Theorem

In this section we shall prove the classical Nagata-Higman theorem for the nilpotency of nil algebras of bounded index. It was first proved in 1953 by Nagata [188] over a field of characteristic 0 and then in 1956 by Higman [127] in the general setup. Much later it was discovered that this theorem was first established in 1943 by Dubnov and Ivanov [99] but their paper was overlooked by the mathematical community. It turns out that this theorem has many applications not only in the theory of PI-algebras but also in (both commutative and noncommutative) invariant theory and structure theory of rings. We refer to the survey article [107] or the book [108] by Formanek for relations with invariant theory. Of course, since we consider nil algebras, we assume that the algebras are nonunitary.

Exercise 8.3.1 Show that, if $\mathrm{char} K \neq 2$, then the identity $x^2 = 0$ implies the identity $x_1x_2x_3 = 0$.

Hint. There are several possible solutions. Try the following. Linearizing $x^2 = 0$ we obtain

$$e(x_1, x_2) = x_1x_2 + x_2x_1.$$

Consider the identities

$$e(x_1x_2, x_3) = (x_1x_2)x_3 + x_3(x_1x_2) = 0,$$

$$e(x_2x_3, x_1) = (x_2x_3)x_1 + x_1(x_2x_3) = 0,$$

$$e(x_3x_1, x_2) = (x_3x_1)x_2 + x_2(x_3x_1) = 0$$

as a linear homogeneous system with unknowns $x_1x_2x_3$, $x_2x_3x_1$ and $x_3x_1x_2$. Show that the determinant of this system is different from 0, hence the only solution of the system is

$$x_1x_2x_3 = x_2x_3x_1 = x_3x_1x_2 = 0.$$

Theorem 8.3.2 (Dubnov-Ivanov-Nagata-Higman [99, 188, 127]) *Let R be a nonunitary associative algebra over a field K of characteristic 0 and let R satisfy the polynomial identity $x^k = 0$. There exists an integer $d = d(k)$ depending on k only such that R is nilpotent of class d, i.e. R satisfies the polynomial identity $x_1 \dots x_d = 0$.*

Proof. We shall work modulo the identity $x^k = 0$, i.e. in the relatively free algebra $F(\mathfrak{N})$ of the variety of nonunitary algebras $\mathfrak{N}$ defined by the polynomial identity $x^k = 0$. The partial linearization of $x^k = 0$ is

$$f(x, y) = x^{k-1}y + x^{k-2}yx + \ldots + xyx^{k-2} + yx^{k-1} = 0.$$

Simple algebraic manipulations give

$$f(x,yz^j)z^{k-j-1}=x^{k-1}yz^{k-1}+x^{k-2}yz^jxz^{k-j-1}+$$

$$+\ldots+xyz^jx^{k-2}z^{k-j-1}+yz^jx^{k-1}z^{k-j-1}=0,$$

$$\sum_{j=0}^{k-1}f(x,yz^j)z^{k-j-1}=kx^{k-1}yz^{k-1}+\sum_{i=0}^{k-2}x^iyf(z,x^{k-i-1})=0.$$

Since $f=0$, we obtain

$$x^{k-1}yz^{k-1}=0.$$

By induction, the identity $x^{k-1}=0$ implies nilpotency. Hence, for some $d=d(k-1)$

$$x_1\ldots x_d=\sum_i a_ib_i^{k-1}c_i,\ x_{d+2}\ldots x_{2d+1}=\sum_j u_jv_j^{k-1}w_j,$$

where $b_i,v_j\in F(\mathfrak{N})$ and a_i,c_i,u_j,w_j are constants in K or polynomials in $F(\mathfrak{N})$. Therefore

$$x_1\ldots x_dx_{d+1}x_{d+2}\ldots x_{2d+1}=\sum_{i,j}a_i(b_i^{k-1}(c_ix_{d+1}u_j)v_j^{k-1})w_j=0$$

and the algebra $F(\mathfrak{N})$ is nilpotent.

Exercise 8.3.3 Show that in the Nagata-Higman theorem the class of nilpotency $d(k)$ is bounded by 2^k-1.

Hint. For $k=1$ one obviously has $d(1)=1$. See that in the proof of the Nagata-Higman theorem given above $d(k)\leq 2d(k-1)+1$.

Exercise 8.3.4 Show that the proof of the Nagata-Higman theorem works also when $\mathrm{char}K=p>k$.

Hint. Since the field has more than k elements, we can apply Vandermonde arguments and obtain that the partial linearization $f(x,y)=0$ is a consequence of $x^k=0$. Since $\mathrm{char}K=p>k$, the identity $kx^{k-1}yz^{k-1}=0$ implies $x^{k-1}yz^{k-1}=0$.

Exercise 8.3.5 Show that the condition $\mathrm{char}K=p>k$ in Exercise 8.3.4 is essential. Construct an algebra which satisfies the identity $x^p=0$ and is not nilpotent.

Hint. Consider the polynomial algebra $\mathbb{F}_p[X]$ in infinitely many variables modulo the ideal generated by all x_i^p, $x_i\in X$. Show that the (nonunitary) subalgebra generated by X satisfies $x^p=0$ but is not nilpotent.

For many purposes it is important to know the exact value of the class of nilpotency $d = d(k)$ in the Nagata-Higman theorem. The upper bound given in the proof of Higman [127] is $d(k) \leq 2^k - 1$ (see Exercise 8.3.3). The best known upper bound is due to Razmyslov [219]. Using his method to study polynomial identities of algebras (see his book [221]), Razmyslov proved that the polynomial identities of $M_k(K)$ follow from $x^k = 0$ and obtained the bound $d(k) \leq k^2$. On the other hand, Kuzmin [160] showed that there exists an algebra which is nil of index k and does not satisfy the identity $x_1 \dots x_m = 0$, where $m = \frac{1}{2}k(k+1) - 1$. In this way

$$\frac{k(k+1)}{2} \leq d(k) \leq k^2.$$

Problem 8.3.6 Find the exact value $d(k)$ of the class of nilpotency of nil algebras of index k over a field of characteristic 0.

Conjecture 8.3.7 (Kuzmin [160]) The exact value $d(k)$ of the class of nilpotency of nil algebras of index k over a field of characteristic 0 is

$$d(k) = \frac{k(k+1)}{2}.$$

The only values of $d(k)$ are known for $k \leq 4$: For $k \leq 3$ they were obtained by Dubnov [98] in 1935:

$$d(1) = 1, d(2) = 3, d(3) = 6,$$

and this agrees with the conjecture of Kuzmin. Very recently, in 1993, Vaughan-Lee [261] proved that

$$d(4) = 10,$$

confirming the conjecture of Kuzmin also for $k = 4$.

We complete this section with an example of Kemer showing that in positive characteristic all polynomial identities of an algebra may not depend "too much" on the multilinear polynomial identities.

Exercise 8.3.8 (Kemer [143]) Let $\mathrm{char} K = p > 0$ and let R be a PI-algebra with a basis as a vector space $\{r_i \mid i \in I\}$. Consider the algebra

$$C = K[\xi_{ij} \mid i \in I, j = 1, 2, \dots]/(\xi_{ij}^2 \mid i \in I, j = 1, 2, \dots),$$

i.e. the polynomial algebra modulo the ideal generated by all squares of the variables. Let S be the nonunitary subalgebra of $C \otimes_K R$ generated by

$$\{\xi_{ij} \otimes r_i \mid i \in I, j = 1, 2, \dots\}.$$

Show that the algebras R and S satisfy the same multilinear polynomial identities and every element of S is nil.

Hint. A multilinear polynomial identity $f(x_1,\dots,x_n)=0$ holds for R if and only if $f(r_{i_1},\dots,r_{i_n})=0$ for all basis elements $r_{i_1},\dots,r_{i_n}$ (including repetitions) and this is if and only if

$$f(\xi_{i_1 1}\otimes r_{i_1},\dots,\xi_{i_n n}\otimes r_{i_n})=0.$$

8.4 The Theory of Kemer

It is well known that the Jacobson radical of finite dimensional algebras and finitely generated commutative algebras is nilpotent. The theorem of Razmyslov-Kemer-Braun states that the same holds for finitely generated PI-algebras. (Again PI-algebras enjoy a nice property as nilpotency of the radical.) Razmyslov [220] proved that over a field of characteristic 0 the Capelli identity implies the nilpotency of the radical. Kemer [140] established that every finitely generated PI-algebra satisfies some Capelli identity and in this way the result was obtained over a field of characteristic 0. Finally, Braun [38] proved the theorem in the maximally general case of finitely generated PI-algebras over a commutative noetherian ring K. Here we prove another theorem of Kemer [137] which states that the standard identity implies the Capelli identity. The proof of the theorem gives also some (very slight) idea about the methods developed by Kemer in his structure theory of T-ideals discussed in the end of this chapter. We divide the proof in several exercises with detailed hints, but leaving also some work to the reader. Below we assume that $\mathrm{char} K=0$.

Let $X\cup Y=\{x_1,x_2,\dots,y_1,y_2,\dots\}$ and let $K\langle X\cup Y\rangle$ be the free associative unitary algebra freely generated by $X\cup Y$. We consider the canonical homomorphism

$$\nu: K\langle X\cup Y\rangle \longrightarrow K\langle X\cup Y\rangle/(x_iwx_j+x_jwx_i \mid w\in K\langle Y\rangle, i,j=1,2,\dots).$$

We denote $\nu(x_i)=e_i$ and $\nu(y_i)=y_i$, $i=1,2,\dots$ Hence the algebra $A=\nu(K\langle X\cup Y\rangle)$ is generated by $e_1,e_2,\dots$ and $y_1,y_2,\dots$, satisfies the relations

$$e_iwe_j=-e_jwe_i,\ i,j=1,2,\dots,w\in K\langle Y\rangle,$$

and may be considered as a generalization of the Grassmann algebra.

Exercise 8.4.1 Show that the algebra A has a basis as a vector space

$$w_1e_{i_1}w_2e_{i_2}w_3\dots w_pe_{i_p}w_{p+1},\ i_1<i_2<\dots<i_p,$$

where $w_1,\dots,w_{p+1}$ are monomials in $K\langle Y\rangle$, $p=0,1,2,\dots$

Hint. Use arguments as in the proof of Poincaré-Birkhoff-Witt Theorem 1.3.2. (Compare with Exercise 1.3.11.) Pay special attention to the analogue of Lemma 1.3.3. Usually the uniqueness of the reduced form is based on the so called Diamond Lemma. See the paper by Bergman [33] for the Diamond Lemma and its applications in ring theory.

Exercise 8.4.2 Let

$$d_n(x_1,\ldots,x_n;y_1,\ldots,y_{n+1})=\sum_{\sigma\in S_n}(\mathrm{sign}\sigma)y_1x_{\sigma(1)}y_2\ldots y_nx_{\sigma(n)}y_{n+1}$$

be the Capelli identity in n skew symmetric variables and let

$$D_n=\mathrm{span}\{d_n(x_1,\ldots,x_n;h_1,\ldots,h_{n+1})\mid h_i\in K\langle Y\rangle\},$$

$$G_n=\mathrm{span}\{h_1e_1h_2e_2\ldots h_ne_nh_{n+1}\mid h_i\in K\langle Y\rangle\}.$$

Show that the map ϕ_n defined by

$$\phi_n:d_n(x_1,\ldots,x_n;h_1,\ldots,h_{n+1})\longrightarrow h_1e_1h_2e_2\ldots h_ne_nh_{n+1}$$

induces an isomorphism of vector spaces $\phi_n:D_n\longrightarrow G_n$.

Hint. Show that D_n and G_n have bases respectively

$$\{d_n(x_1,\ldots,x_n;w_1,\ldots,w_{n+1})\},\ \{w_1e_2w_2e_2\ldots w_ne_nw_{n+1}\}$$

where $w_1,\ldots,w_{n+1}$ run on the set of all monomials in $K\langle Y\rangle$.

Exercise 8.4.3 Let U be a T-ideal of $K\langle X\cup Y\rangle$ and let $U(A)=\nu(U)$ be its image in A. Show that

$$\phi_n(U\cap D_n)=U(A)\cap G_n.$$

Hint. If

$$f(x_1,\ldots,x_n;y_1,\ldots,y_m)=\sum_h\alpha_hd_n(x_1,\ldots,x_n;h_1,\ldots,h_{n+1})\in U\cap D_n,$$

$\alpha_h\in K$, $h_i\in K\langle Y\rangle$, then $\nu(f)\in U(A)$ and

$$\nu(f)=f(e_1,\ldots,e_n;y_1,\ldots,y_m)=\sum_h\alpha_hd_n(e_1,\ldots,e_n;h_1,\ldots,h_{n+1})=$$

$$=n!\sum_h\alpha_hh_1e_1h_2e_2\ldots h_ne_nh_{n+1}=n!\phi_n(f)\in G_n.$$

If $\nu(f)\in U(A)\cap G_n$, then

$$\sum_{\sigma\in S_n}(\mathrm{sign}\sigma)f(e_{\sigma(1)},\ldots,e_{\sigma(n)},y_1,\ldots,y_m)=$$

$$= n!\phi_n \left(\sum_{\sigma \in S_n} (\mathrm{sign}\sigma) f(x_{\sigma(1)}, \ldots, x_{\sigma(n)}, y_1, \ldots, y_m) \right),$$

$$\sum_{\sigma \in S_n} (\mathrm{sign}\sigma) f(x_{\sigma(1)}, \ldots, x_{\sigma(n)}, y_1, \ldots, y_m) \in U \cap D_n .$$

Exercise 8.4.4 Let U be a T-ideal of $K\langle X \cup Y\rangle$. Show that:

(i) The standard polynomial $s_k(x_1, \ldots, x_k)$ belongs to U if and only if the product $e_1 \ldots e_k$ belongs to $U(A)$.

(ii) The Capelli polynomial $d_n(x_1, \ldots, x_n; y_1, \ldots, y_{n+1})$ belongs to U if and only if $Ae_1Ae_2A \ldots Ae_nA$ is a subset of $U(A)$.

Hint. Use Exercise 8.4.3 and the equations

$$s_k(e_1, \ldots, e_k) = k! e_1 \ldots e_k ,$$

$$d_n(e_1, \ldots, e_n; w_1, \ldots, w_{n+1}) = n! w_1 e_1 w_2 e_2 \ldots w_n e_n w_{n+1} .$$

Theorem 8.4.5 (Kemer [137]) *Over a field of characteristic* 0 *the standard identity*

$$s_k(x_1, \ldots, x_k) = 0$$

implies some Capelli identity

$$d_n(x_1, \ldots, x_n; y_1, \ldots, y_{n+1}) = \sum_{\sigma \in S_n} (\mathrm{sign}\sigma) y_1 x_{\sigma(1)} y_2 \ldots y_n x_{\sigma(n)} y_{n+1} = 0.$$

Proof. We use the same notation as above. Let U be the T-ideal generated in $K\langle X \cup Y\rangle$ by the standard polynomial $s_k(x_1, \ldots, x_k)$. Replacing, if necessary, k by $k+1$, we may assume that $k = 2p$ is even. Let I_q be the vector subspace of A spanned by all products $e_{i_1} \ldots e_{i_q}$, $i_1 < \ldots < i_q$. By Exercise 8.4.4 (ii) it is sufficient to show that, for some n,

$$(AI_1A)^n \subseteq U(A) = AI_{2p}A + U(A).$$

If, for every $q = 1, 2, \ldots, 2p-1$, there exists a t such that

$$(I_qA)^t \subseteq AI_{q+1}A + U(A),$$

then

$$(AI_1A)^t = A(I_1A)^t \subseteq AI_2A + U(A),$$

$$((AI_1A)^t)^t \subseteq (AI_2A)^t + U(A) = A(I_2A)^t + U(A) \subseteq AI_3A + U(A), \ldots,$$

$$(AI_1A)^{t^{2p-1}} \subseteq AI_{2p}A + U(A) = U(A).$$

We shall show that the subalgebra of A generated by I_qA modulo the ideal $AI_{q+1}A + U(A)$, is nil of index bounded by $p+1$. By Nagata-Higman Theorem

8.3.2, this implies that $(I_qA)^t \subseteq AI_{q+1}A + U(A)$ for some $t = t(q)$ and the proof will be completed. Let $b_1, \dots, b_{p+1} \in I_q$, $v_1, \dots, v_p \in A$. We may assume that b_i are monomials of degree q in $e_1, e_2, \dots$ For $q = 1$ we obtain

$$s_{2p}(b_1, v_1, b_2, v_2, \dots, b_p, v_p) =$$

$$= \sum_{\sigma \in S_p} \sum_{\tau \in S_p} (\text{sign}\sigma\tau) b_{\sigma(1)} v_{\tau(1)} b_{\sigma(2)} v_{\tau(2)} \dots b_{\sigma(p)} v_{\tau(p)} +$$

$$+(-1)^p \sum_{\sigma \in S_p} \sum_{\tau \in S_p} (\text{sign}\sigma\tau) v_{\tau(1)} b_{\sigma(1)} v_{\tau(2)} b_{\sigma(2)} \dots v_{\tau(p)} b_{\sigma(p)} + \dots \in U(A),$$

where $\dots$ denotes the sum of all products with two consecutive b_ib_j and $\dots \in AI_2A$. Multiplying from the right by b_{p+1}, we obtain that

$$u = \sum_{\sigma \in S_p} \sum_{\tau \in S_p} (\text{sign}\sigma\tau) b_{\sigma(1)} v_{\tau(1)} b_{\sigma(2)} v_{\tau(2)} \dots b_{\sigma(p)} v_{\tau(p)} b_{p+1}$$

is in $AI_2A + U(A)$. Recall that $q = 1$, i.e. the b_i's are equal to some e_{j_i}. We use the relations

$$b_2 w b_1 = -b_1 w b_2,$$

which hold for all $w \in A$. (In the definition of A we required these relations for $w \in K\langle Y \rangle$ only. Why they hold for any $w \in A$?) We obtain for the above expression of $u \in AI_2A + U(A)$ that

$$u = p! \sum_{\sigma \in S_p} (b_{\sigma(1)} v_{\sigma(1)})(b_{\sigma(2)} v_{\sigma(2)}) \dots (b_{\sigma(p)} v_{\sigma(p)}) b_{p+1} =$$

$$= h(b_1v_1, \dots, b_pv_p) b_{p+1} \in AI_2A + U(A),$$

where

$$h(x_1, \dots, x_p) = \sum_{\sigma \in S_p} x_{\sigma(1)} \dots x_{\sigma(p)}$$

is the complete linearization of the polynomial x^p. Every expression $(z_1 + \dots + z_r)^p$ can be written as a linear combination of elements, obtained by substitutions in $h(x_1, \dots, x_p)$. Hence

$$(b_1v_1 + b_2v_2 + \dots + b_rv_r)^{p+1} \in \sum_{i=1}^{r} (b_1v_1 + b_2v_2 + \dots + b_rv_r)^p b_i A \subseteq AI_2A + U(A).$$

Now, if $q > 1$, we write each b_i in the form $b_i = e_{j_i} b_i'$. We consider the standard polynomial

$$s_{2p}(e_{j_1}, (b_1'v_1), e_{j_2}, (b_2'v_2), \dots, e_{j_p}, (b_p'v_p)) \in U(A)$$

and apply considerations similar to those for $q = 1$ in order to obtain

$$(b_1v_1 + b_2v_2 + \dots + b_rv_r)^{p+1} \in AI_{q+1}A + U(A),\ b_i \in I_q, v_i \in A.$$

Finally, we give a very brief account of some results of Kemer [141] which show that the T-ideals of $K\langle X\rangle$ behave as the ideals of the polynomial algebras in several variables and show the importance of the matrix and Grassmann algebras for the theory of PI-algebras. We assume that the field is of characteristic 0.

Definition 8.4.6 The T-ideal S of $K\langle X\rangle$ is called *T-semiprime* if any T-ideal U with $U^n \subseteq S$ for some n is included in S, i.e. $U \subseteq S$. The T-ideal P is *T-prime* if the inclusion $U_1U_2 \subseteq P$ for some T-ideals U_1 and U_2 implies $U_1 \subseteq P$ or $U_2 \subseteq P$.

Clearly, the above definition is similar to the usual definition of semiprime and prime ideals, but instead of ordinary ideals, we consider T-ideals only.

Exercise 8.4.7 Show that every T-prime T-ideal is T-semiprime.

Hint. If P is T-prime and $U^n \subset P$ for some T-ideal U, then $U^n = UU^{n-1}$ and $U \subset P$ or $U^{n-1} \subset P$.

Let E be the Grassmann algebra of an infinite dimensional vector space. It has a natural $\mathbb{Z}_2$-grading $E = E_0 \oplus E_1$, which we have already used in the Rosset proof of Amitsur-Levitzki Theorem 7.1.8. Let p, q be positive integers, $p \geq q$. Consider the vector subspace of $M_{p+q}(E)$

$$M_{p,q} = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid a \in M_p(E_0), b \in M_{p\times q}(E_1), c \in M_{q\times p}(E_1), d \in M_q(E_0) \right\}$$

where $M_{p\times q}(E_1)$ is the vector space of all $p \times q$ matrices with entries from E_1, similarly for the other sets of matrices. For example,

$$\begin{pmatrix} e_3e_1 & e_2e_4 & e_1 \\ 1 & e_3e_1e_2e_4 & e_1e_4e_2 \\ e_2 & e_2 & e_1e_2 \end{pmatrix}$$

is an element of $M_{2,1}$ with

$$a = \begin{pmatrix} e_3e_1 & e_2e_4 \\ 1 & e_3e_1e_2e_4 \end{pmatrix} \in M_2(E_0),\ b = \begin{pmatrix} e_1 \\ e_1e_4e_2 \end{pmatrix} \in M_{2\times 1}(E_1),$$

$$c = \begin{pmatrix} e_2 & e_2 \end{pmatrix} \in M_{1\times 2}(E_1),\ d = \begin{pmatrix} e_1e_2 \end{pmatrix} \in M_1(E_0).$$

Exercise 8.4.8 Show that $M_{p,q}$ is an algebra.

Exercise 8.4.9 Show that the T-ideals $T(M_k(K))$, $T(M_k(E))$ and $T(M_{p,q})$ are T-prime for all $k \geq 1$ and $p \geq q \geq 1$.

Hint. Let $P = T(R)$, where R is one of the algebras $M_k(K)$, $M_k(E)$ and $M_{p,q}$. Let U_1 and U_2 be T-ideals which are not contained in P. Choose $f_1 \in U_1$, $f_2 \in U_2$, $f_1, f_2 \notin P$. We may assume that $f_1 = f_1(x_1, \ldots, x_m)$, $f_2 = f_2(x_{m+2}, \ldots, x_n)$. (Why?) Fix $r_1, \ldots, r_m, s_{m+2}, \ldots, s_n \in R$ with $f_1(r_1, \ldots, r_m) \neq 0$, $f_2(s_{m+2}, \ldots, s_n) \neq 0$. (If $R = M_k(E)$ or $R = M_{p,q}$, choose r_i, s_j in such a way that they depend on disjoint sets of generators $\{e_1, \ldots, e_a\}$ and $\{e_{a+1}, \ldots, e_b\}$ of E.) Find a matrix $r_{m+1} \in R$ satisfying $f_1(r_1, \ldots, r_m)r_{m+1}f_2(s_{m+2}, \ldots, s_n) \neq 0$. Hence U_1U_2 is not contained in P.

One of the main theorems in the structure theory of T-ideals developed by Kemer gives a list of all T-semiprime and T-prime T-ideals and a description of the T-ideals similar to that of the ideals in the polynomial algebra. An exposition of his theory can be found in his book [142] or in the original paper [141].

Theorem 8.4.10 (Kemer [141]) (i) *For every T-ideal U of $K\langle X\rangle$ there exists a T-semiprime T-ideal S and a positive integer n such that*

$$S \supseteq U \supseteq S^n.$$

(ii) *Every T-semiprime T-ideal S is an intersection of a finite number of T-prime T-ideals $Q_1, \ldots, Q_m$*

$$S = Q_1 \cap \ldots \cap Q_m.$$

(iii) *A T-ideal P is T-prime if and only if P coincides with one of the following T-ideals:*

$$T(M_k(K)), T(M_k(E)), T(M_{p,q}), (0), K\langle X\rangle,$$

where $M_k(E)$ is the $k \times k$ matrix algebra with entries from the Grassmann algebra E.

9. Finitely Generated PI-Algebras

In this chapter we apply combinatorial theory of words to the study of finitely generated PI-algebras. After discussions on the Burnside problem in group theory and the Kurosch problem in ring theory we present the methods of Shirshov based on the notion of height of words. It leads to purely combinatorial proofs of the theorems of Levitzki and Kaplansky which give positive answers to the Kurosch problem for PI-algebras. The approach of Shirshov brings also some quantitative results as the theorem of Berele that the Gelfand-Kirillov dimension of a finitely generated PI-algebra is finite, which makes the finitely generated PI-algebras close to finitely generated commutative algebras. On the other hand, we give some exotic examples of PI-algebras with prescribed Gelfand-Kirillov dimension. The main results hold both for unitary and nonunitary algebras and over any field.

9.1 The Problems of Burnside and Kurosch

In 1902 Burnside [44] asked his famous problem which has been one of the main driving forces of the theory of infinite groups for almost 90 years. There are three different versions of the Burnside problem.

Problem 9.1.1 (Burnside Problem) Let G be a finitely generated group such that every element of G is of finite order.

(i) Is the group G finite?

(ii) If there exists a positive integer k such that every element g of G satisfies the equation $g^k = 1$, is G finite? (Restatement in the language of varieties of groups: Is the variety of groups, defined by the identity $x^k = 1$, locally finite?)

(iii) (Restricted Burnside Problem) If G is finite, generated by m elements and satisfies the condition $g^k = 1$ for all $g \in G$, does there exist a bound for the order of G in terms of m and k? (In the language of varieties: Is the variety generated by all finite groups of exponent k locally finite?)

It is very interesting that the negative solution of Problem 9.1.1 (i) was obtained by Golod [117] using ring theoretic constructions. Now there are many counterexamples based on different approaches. Maybe the simplest ones are of Grigorchuk [119] who constructed a finitely generated 2-group which is not finite and of N.D. Gupta and Sidki [121] based on graph theory. Another easy counterexample of Aleshin [7] involving automata theory is given in the third and fourth Russian editions of the book by Kargapolov and Merzlyakov [135].

The negative solution of Problem 9.1.1 (ii) was obtained by Novikov and Adyan [195] for k odd and sufficiently large (in the book by Adyan [6] the result is given for odd $k \geq 665$). Other counterexamples were discovered by Olshanskii using his geometric approach to groups presented by generators and defining relations. In particular, for every prime $p > 10^{10}$ Olshanskii constructed the so called Tarski-Olshanskii monster of exponent p. This is an infinite group in each all proper nontrivial subgroups are of order p.

For many small k the answer to Burnside Problem 9.1.1 (ii) is still unknown. For example, it is still an open problem whether every 2-generated group satisfying the identity $x^5 = 1$ is finite.

Finally, using methods of Lie algebras satisfying the Engel condition (i.e. a special polynomial identity) Kostrikin (see his book [154]) proved that for p prime the order of the m-generated finite groups of exponent p is bounded. Later the problem (also in the affirmative and also using a ring theory approach) was solved by Zelmanov [269, 270] for any k.

The restatement of the Burnside problem for algebras is known as the Kurosch problem.

Problem 9.1.2 (Kurosch Problem [159]) Let R be a finitely generated associative algebra such that every element of R is algebraic.

(i) Is the algebra R finite dimensional?

(i$'$) If R is nil, is it nilpotent?

(ii) If every element of R is algebraic of bounded degree, is R finite dimensional?

(ii$'$) If R is nil of bounded index, is it nilpotent?

For Lie algebras the "nil" condition is replaced by the Engel condition.

Problem 9.1.3 (i) Let G be a finitely generated Engel Lie algebra, i.e. for every two elements $g, h \in G$ there exists an $n = n(g,h) > 0$ such that $g\mathrm{ad}^n h = 0$. Is G nilpotent?

(ii) If the Lie algebra G satisfies an Engel condition of bounded degree (i.e. the polynomial identity $[x, y, \ldots, y] = 0$), is G (locally) nilpotent?

The negative answer to Problem 9.1.2 (i$'$) (and hence also to (i)) was given by Golod and Shafarevich [118]. They used some quantitative approach to construct a series of counterexamples, which serve also as counterexamples to Problem 9.1.3 (i). Concerning Problem 9.1.2 (ii$'$) we have seen that, if the characteristic of the field is 0 or sufficiently large, Nagata-Higman Theorem

8.3.2 gives that the algebra is nilpotent even without the condition that it is finitely generated.

The nil algebras of bounded index satisfy the polynomial identity $x^k = 0$ for some k. Similarly, if all elements of the algebra are algebraic of bounded degree k, then $1, a, a^2, \dots, a^k$ are linearly depended for any $a \in R$ and this implies that R satisfies the identity of algebraicity, as the $k \times k$ matrix algebra. Since we have already seen (for example in the previous Chapter 8) that the class of all PI-algebras has some nice properties, one may expect that the Kurosch problem has a positive solution for PI-algebras.

Problem 9.1.4 (The Kurosch problem for PI-algebras) If R is a finitely generated PI-algebra and every element of R is algebraic (or nil), is R finite dimensional (or nilpotent)?

For nil PI-algebras the problem was answered (in the affirmative) by Levitzki [172] and the general problem for algebraic PI-algebras was solved, also positively, by Kaplansky [132]. Both the proofs involve structure theory of rings and can be found in the book by Herstein [126].

We shall mention also three results on Lie algebras related with Problem 9.1.3 (ii). Kostrikin (see [154]) proved that the Lie algebras satisfying the Engel condition $x\mathrm{ad}^k y = 0$ are locally nilpotent, where $\mathrm{char}K = 0$ or $\mathrm{char}K = p \geq k$. Razmyslov [216] constructed an example of an infinitely generated Lie algebra satisfying the Engel condition $x\mathrm{ad}^{p-2} y = 0$ over a field of characteristic $p > 3$, which is not nilpotent. Finally, Zelmanov [268] established that the Engel identity implies the nilpotency of the algebra if $\mathrm{char}K = 0$. One may see an exposition of the relations between similar problems in groups, associative algebras and Lie algebras in the book by Zelmanov [271].

9.2 The Shirshov Theorem

In this section we consider the famous Shirshov theorem [238]. There are several expositions of this theorem in the literature. For example, a version of the original proof of Shirshov can be found in the book by Zhevlakov, Slinko, Shestakov and Shirshov [272]. We give the proof of A.Ya. Belov (see the survey article by Amitsur and Small [13]).

We fix the number $m > 1$ of the generators of the free associative algebra. Let $W = \langle x_1, x_2, \dots, x_m \rangle$ be the set of all monomials (words) in $K\langle X_m \rangle$. It is called the *free unitary semigroup of rank m* and is the free object in the class of all semigroups with unity. For $w = x_{i_1} x_{i_2} \dots x_{i_n} \in W$ we denote by $|w|$ the *length* (or the degree) of w.

Definition 9.2.1 We introduce a partial lexicographic ordering on W assuming that $x_1 < x_2 < \dots < x_m$, and then extending it on W in the following way:

$$x_{i_1}\dots x_{i_p} > x_{j_1}\dots x_{j_q}$$

if and only if $i_1 = j_1, \dots, i_k = j_k, i_{k+1} > j_{k+1}$ for some $k \geq 0$. Two words u and v are *incomparable* if one of them is a *beginning* of the other (i.e. $u = vw$ for some $w \in W$, $w \neq 1$). A finite subset of W is called *incomparable* if it contains two incomparable words.

Definition 9.2.2 The word $w \in W$ is called *d-decomposable* if it can be written in the form

$$w = w_0w_1\dots w_dw_{d+1},$$

(some of the words w_0 and w_{d+1} may be empty) and

$$w_0w_1\dots w_dw_{d+1} > w_0w_{\sigma(1)}\dots w_{\sigma(d)}w_{d+1}$$

for every nontrivial permutation $\sigma \in S_d$.

For example,

$$w = x_2x_1x_3x_2x_1x_2x_3x_1x_2x_4x_1 = (x_2x_1)(x_3x_2x_1x_2)(x_3x_1x_2)(x_4x_1)$$

is a 2-decomposition, with $w_0 = x_2x_1$ and $w_3 = x_4x_1$, because

$$(x_2x_1)(x_3x_2x_1x_2)(x_3x_1x_2)(x_4x_1) >$$

$$> (x_2x_1)(x_3x_1x_2)(x_3x_2x_1x_2)(x_4x_1) = w_0w_2w_1w_3.$$

This word has also a 4-decomposition

$$w = (x_2x_1)(x_3x_2x_1x_2)(x_3x_1)(x_2x_4)(x_1),\ w_0 = x_2x_1,\ w_5 = 1.$$

(Prove that this is really a 4-decomposition!)

Lemma 9.2.3 *If $w \in W$ has the presentation $w = pq = rp$ for some nonempty words p, q and r, then w has the form a^n or $abab\dots aba = (ab)^{n-1}a$ for some $a, b \neq 1$ and $n > 1$.*

Proof. (i) If $|p| < |r|$, then $r = pb$ and hence $w = pbp$, i.e. $w = aba$ for $a = p$.

(ii) If $|r| \leq |p|$, then $p = rp_1$. Hence $w = rp_1q = rrp_1$ and $p_1q = rp_1$. If $|r| \leq |p_1|$, then $p_1 = rp_2$ and $p_2q = rp_2$. We continue this process until we obtain $p_k = rp_{k+1}$ and $p_{k+1}q = rp_{k+1}$, where $|p_{k+1}| < |r|$. In this way we obtain:

(a) If $p_{k+1} = 1$, then $p = r^{k+1}$ and $w = r^{k+2} = a^{k+2}$ for $a = r$.

(b) If $p_{k+1} \neq 1$, then $p_{k+1}q = rp_{k+1}$ and $|p_{k+1}| < |r|$. From the case (i) we obtain for $a = p_{k+1}$, $r = ab$, that $p = r^{k+1}p_{k+1}$ and

$$w = r_{k+2}p_{k+1} = (ab)^{k+2}a.$$

This completes the proof of the lemma.

Lemma 9.2.4 *Let*

$$v = w_0ww_1ww_2\dots w_{d-1}ww_d$$

be a word such that the subword w has d different comparable subwords. Then the word v is d-decomposable.

Proof. Let $w = a_iv_ib_i$, $i = 1,\dots,d$, and $v_1 > v_2 > \dots > v_d$. Then v has the following d-composition

$$v = (w_0a_1)(v_1b_1w_1a_2)(v_2b_2w_2a_3)\dots(v_{d-1}b_{d-1}w_{d-1}a_d)(v_db_d)w_{d+1},$$

because $t_1 > t_2 > \dots > t_d$, where $t_i = v_ib_iw_ia_{i+1}$, $i = 1,\dots,d-1$, and $t_d = v_db_d$.

Example 9.2.5 Let p and q be comparable words. Then the word $w = p^{d-1}q$ contains d comparable subwords. If $p > q$, then

$$p^{d-1}q > p^{d-2}q > \dots > pq > q.$$

If $p < q$, then

$$p^{d-1}q < p^{d-2}q < \dots < pq < q.$$

Definition 9.2.6 Let $w \in W$ and $|w| \geq d$. We write

$$w = w_1 = e_2w_2 = \dots = e_dw_d,$$

where e_i is a beginning of w and $|e_i| = i-1$. Then $|w_i| = |w| - i + 1$. We call the words $w_1, w_2, \dots, w_d$ *d-ends* of w.

Lemma 9.2.7 *Let $|w| \geq d$ and let the d-ends of w be incomparable. Then $w = ab^tc$, where $|a| + |b| < d$, $t \geq 1$ and either $c = 1$ or c is a beginning of b. If $|w| \geq dk$, then $t \geq k$, and, in particular, w contains a subword b^k with $|b| < d$.*

Proof. Let w_i and w_j be two incomparable d-ends of w, $i < j$. Therefore $w = aw_i$ and $w = abw_j$, hence $w_i = bw_j$. Since w_i and w_j are incomparable, this means that $w_i = w_ju$ and by Lemma 9.2.3 ($w_i = bw_j = w_ju$) we obtain $w_i = b^tc$, where either $c = 1$ or c is a beginning of b, i.e. $w = ab^tc$. Clearly, $|a| + |b| = j - 1 < d$. If $dk \leq |w|$, then

$$dk \leq |a| + |b| + (t-1)|b| + |c| < d + (t-1)|b| + |c| < d + t|b| < (t+1)d$$

and $k \leq t$.

Lemma 9.2.8 *Let $w \in W$ and $|w| = kd$. If w does not contain a subword b^k, $|b| < d$, then $w = vu$, the d-ends of v are comparable and v belongs to a finite set of elements S such that the number of the elements $|S|$ of S is bounded by*

$$|S| \leq s(d,k) = \binom{d}{2}(k-1)m^d.$$

Proof. If the d-ends of w are not comparable, then by Lemma 9.2.7, the word w contains a subword b^k, $|b| < d$. Hence the d-ends of w are comparable. Let v be the beginning of w of minimal length such that the d-ends of v are comparable. By the definition of d-ends (see Definition 9.2.6), $|v| \geq d$. Let $v = qx$ for some letter x. Then either $|q| < d$ or the d-ends of q are incomparable (because q is a beginning of w which is shorter than v). In the second case Lemmas 9.2.3 and 9.2.7 give that $q = a(cb)^t c$, where $c, b \neq 1$, $|a|+|b|+|c| < d$ and $t \geq 1$ or $q = ab^t$, where $b \neq 1$, and $|a|+|b| < d$. The case $t \geq k$ is impossible, because w contains $(bc)^t$ and $|bc| = |b| + |c| < d$. Hence $t < k$. We denote by S the set of all words written in one of the following two forms:

(i) $v = a(cb)^t cx$, where $c, b \neq 1$, $1 \leq t \leq k-1$, $|a| + |b| + |c| \leq d-1$ and the letter x is different from the first letter of b.

(ii) $v = ab^t x$, where $b \neq 1$, $1 \leq t \leq k-1$, $|a| + |b| \leq d-1$ and x is a letter different from the first letter of b.

Let us estimate the number of the words of the first kind. (In some places we have inequalities because we are not sure that the corresponding presentation is unique.) Let $l = |a|+|b|+|c|$. Therefore $2 \leq l \leq d-1$. The two integers $|a|$ and $|a| + |c|$ determine $|a|$, $|b|$ and $|c|$. Hence we choose two different integers $|a|$ and $|a| + |c|$ between $0, 1, 2, \ldots, l-1$, and we have $\binom{l}{2}$ possibilities for $|a|$, $|b|$ and $|c|$. For fixed l, the possibilities for v are $\binom{l}{2}m^l(k-1)(m-1)$ (m letters in each of the l positions of a, b and c; $k-1$ for $t = 1, \ldots, k-1$ and $m-1$ letters x different from the first letter of b). The summation for $l = 2, 3, \ldots, d-1$ gives that the number of the words of the first kind is bounded by

$$\sum_{l=2}^{d-1} \binom{l}{2} m^l (k-1)(m-1).$$

Similarly, we obtain that the number of the words of the second kind is bounded by

$$\sum_{l=1}^{d-1} l m^l (k-1)(m-1).$$

Hence

$$|S| \leq \sum_{l=2}^{d-1} \binom{l}{2} m^l (k-1)(m-1) + \sum_{l=1}^{d-1} l m^l (k-1)(m-1) =$$

$$= (k-1)(m-1) \sum_{l=1}^{d-1} \binom{l+1}{2} m^l \leq (k-1)(m-1) \binom{d}{2} \sum_{l=0}^{d-1} m^l \leq (k-1) \binom{d}{2} m^d.$$

The following theorem is the version of Belov of the combinatorial theorem of Shirshov [238] which is the key result in our approach to finitely generated PI-algebras.

Theorem 9.2.9 *Let w be a word in the free semigroup $\langle x_1,\dots,x_m\rangle$, $m>1$, and let k and d be fixed integers with $k\geq d>1$. If the word w is not d-decomposable, then it can be written in the form*

$$w=c_0v_1^{k_1}c_1v_2^{k_2}\dots v_r^{k_r}c_r,$$

where

$$|v_i|<d,\ k_i\geq k,\ r<dm^d,\ \sum_{i=0}^{r}|c_i|\leq d^2ks(d,k)+dk(r+1)\leq\frac{d^4k^2m^d}{2},$$

the words v_i and c_i have no common beginning and, if $c_i=1$, then v_i and v_{i+1} have no common beginning. Here $s(d,k)$ was defined in the previous Lemma 9.2.8.

Proof. If $|w|\leq d^2ks(d,k)$, then we assume that $c_0=w$ and complete the proof. If $|w|>d^2ks(d,k)$, then we may write w in the form

$$w=u_1w_1u_2w_2\dots u_tw_tu_{t+1},$$

where $t=ds(d,k)$, $|w_i|=kd$ and the u_i's are arbitrary. By Lemma 9.2.8, if no w_i contains a subword b^k, $|b|<d$, then $w_i=v_ix_i$, where v_i is in the set S defined in Lemma 9.2.8 and v_i contains d comparable subwords. Since $t=ds(d,k)\geq d|S|$, some v_i appears at least d times in w and by Lemma 9.2.4 the word w is d-decomposable. This contradicts with the assumption that w is not d-decomposable. Therefore some w_i (and hence also w) contains a subword b^k, $|b|<d$. Let c_0 be the shortest beginning of w such that $w=c_0v_1^{k_1}w_1$, where $|v_1|<d$, $k_1\geq k$ and the words v_1 and w_1 have no common beginning. The property that v_1 and w_1 have no common beginning can be always arranged. If we assume that for some presentation $w=c_0v_1^{k_1}w_1$, $|v_1|<d$, $k_1\geq k$, the words v_1 and w_1 have the same beginning, e.g. $v_1=pq$ and $w_1=pr$, where q and r have no more a common beginning, then $w=c_0p(qp)^{k_1}r$ and now qp and r have no common beginning. If w_1 contains a subword b^k, $|b|<d$, then we choose c_1 such that $w_1=c_1v_2^{k_2}w_2$ and c_1 is with minimal possible length. As above, we may assume that v_2 and w_2 have no common beginning. We handle w_2 in the same way as w_1, etc. and obtain the following form of w:

$$w=c_0v_1^{k_1}c_1v_2^{k_2}\dots c_{r-1}v_r^{k_r}c_r,$$

where $|v_i|<d$, $k_i\geq k$, c_i does not contain a subword of the form b^k, $|b|<d$, and either v_i and c_i have no common beginning or, if $c_i=1$, then v_i and v_{i+1} have no common beginning. Hence w contains $r-1$ disjoint subwords $v_i^{d-1}x_i$, $i=1,\dots,r-1$, where x_i is a letter different from the first letter of

v_i. Here we use that $k \geq d$ (and that maybe $c_r = 1$). The number of the different words of the form $v^{d-1}x$, $|v| < d$ and x a letter different from the first letter of v is

$$\sum_{l=1}^{d-1} m^l(m-1) = m^d - m < m^d.$$

If we assume that $r \geq dm^d$, then some word of the form $v^{d-1}x$, where $|v| < d$, and x is not the first letter of v, appears in w at least d times:

$$w = q_0(v^{d-1}x)q_1(v^{d-1}x)q_2 \dots q_{d-1}(v^{d-1}x)q_d.$$

By Example 9.2.5, the word $v^{d-1}x$ contains d comparable subwords and by Lemma 9.2.4 we obtain a d-decomposition of w, which is a contradiction. Hence $r < dm^d$. In order to prove the inequality for $\sum_{i=0}^{r} |c_i|$, let us divide each of the words c_i in several consecutive subwords of length dk and a word of length $< dk$. In this way we obtain

$$p = \sum_{i=0}^{r} \left[\frac{|c_i|}{dk}\right] > \frac{1}{dk}\sum_{i=0}^{r} |c_i| - (r+1)$$

intervals of length dk, where $[\alpha]$ is the integer part of $\alpha \in \mathbb{R}$. Since c_i does not contain subwords of the form b^k, $|b| < d$, by Lemma 9.2.8 each of these intervals has for a beginning an element of the set S with d comparable subwords. If $p \geq ds(d,k)$, then we obtain an element of S, which appears d times as a subword of w. This makes the word w d-decomposable. Since this is impossible, we obtain that $p < ds(d,k)$. Using also the inequality

$$\sum_{i=0}^{r} |c_i| < dk(p + (r+1)),$$

we obtain

$$\sum_{i=0}^{r} |c_i| \leq d^2ks(d,k) + dk(r+1) \leq d^2km^d\left(\frac{(k-1)(d-1)d}{2} + 1\right) \leq$$

$$\leq d^2km^d\left(\frac{(k-1)(d-1)d}{2} + 1\right) \leq \frac{1}{2}d^4k^2m^d.$$

Definition 9.2.10 Let R be an algebra generated by $r_1, \dots, r_m$. Let H be a finite set of words of $r_1, \dots, r_m$. One says that R is *of height h with respect to the set of words H* if h is the minimal integer with the property that, as a vector space, R is spanned by all products

$$u_{i_1}^{k_1} \dots u_{i_t}^{k_t}$$

such that $u_{i_1}, \dots, u_{i_t} \in H$ and $t \leq h$.

Exercise 9.2.11 Show that the height of the (commutative) polynomial algebra $K[x_1, \ldots, x_m]$ with respect to the set of words $H = \{x_1, \ldots, x_m\}$ is equal to m.

Hint. As a vector space $K[x_1, \ldots, x_m]$ is spanned by all products $x_1^{k_1} \ldots x_m^{k_m}$ and the monomials including all m variables (e.g. $x_1 \ldots x_m$) cannot be written as linear combinations of words with less than m different powers of x_i.

Exercise 9.2.12 Let $R = F_m(U_k(K))$ be the relatively free algebra of rank m in the variety generated by the algebra $U_k(K)$ of the $k \times k$ upper triangular matrices over an infinite field K. Show that the height of R with respect to the set $H = \{x_1, \ldots, x_m\}$ is bounded from above by $k(m + 2) - 2$.

Hint. Using the results of Section 5.2, show that R is spanned by all products

$$x_1^{a_1} \ldots x_m^{a_m} [x_{i_1}, x_{j_1}] x_1^{b_1} \ldots x_m^{b_m} [x_{i_2}, x_{j_2}] \ldots [x_{i_p}, x_{j_p}] x_1^{c_1} \ldots x_m^{c_m},$$

where $a_i, b_i, \ldots, c_i \geq 0$ and $p \leq k - 1$. Hence R is also spanned by

$$x_1^{a_1} \ldots x_m^{a_m} x_{s_1} x_{t_1} x_1^{b_1} \ldots x_m^{b_m} \ldots x_{s_p} x_{t_p} x_1^{c_1} \ldots x_m^{c_m},\ p \leq k - 1,$$

involving $\leq mk$ powers of x_i and $2p \leq 2(k - 1)$ first powers of x_{s_j} and x_{t_j}.

Now we prove the theorem of Shirshov [238] for the height of finitely generated PI-algebras.

Shirshov Theorem 9.2.13 *Let R be a PI-algebra generated by m elements $r_1, \ldots, r_m$ and satisfying a polynomial identity of degree $d > 1$. Then R is of finite height with respect to the set of all words $r_{i_1} \ldots r_{i_s}$ of length $s < d$.*

Proof. Our considerations will be similar to those in the proof of Theorem of Regev 8.1.7 for the exponential growth of the codimension sequence of a PI-algebra. We assume that R satisfies a multilinear polynomial identity of the form

$$x_1 \ldots x_d = \sum_{\sigma \in S_d} \alpha_\sigma x_{\sigma(1)} \ldots x_{\sigma(d)},\ \alpha_\sigma \in K,$$

where the summation is on all nontrivial permutations $\sigma \in S_d$. Consider a product $w = r_{i_1} \ldots r_{i_p} \in R$. First, if the word w is d-decomposable, then we can write it as a product of $d + 2$ subwords and

$$w = w_0 w_1 \ldots w_d w_{d+1} > w_0 w_{\sigma(1)} \ldots w_{\sigma(d)} w_{d+1}$$

for any nontrivial permutation $\sigma \in S_d$. Then we apply the polynomial identity

$$w_0(w_1 \ldots w_d) w_{d+1} = \sum_{\sigma \in S_d} \alpha_\sigma w_0 (w_{\sigma(1)} \ldots w_{\sigma(d)}) w_{d+1}.$$

We obtain that w is a linear combination of words which are lower in the lexicographic ordering and continue the process until we obtain that all elements of R are linear combinations of words in $r_1, \ldots, r_m$, which are not d-decomposable. Now we fix the integer $k = d$. By Theorem 9.2.9, if the word is not d-decomposable, then it can be written in the form

$$w = c_0 v_1^{k_1} c_1 v_2^{k_2} \ldots v_t^{k_t} c_t,$$

where

$$|v_i| < d,\ k_i \geq k = d,\ t < dm^d,\ \sum_{i=0}^{t} |c_i| \leq \frac{d^4 k^2 m^d}{2} = \frac{d^6 m^d}{2}.$$

Considering the word c_i as a product $c_i = u_{j_1} \ldots u_{j_q}$ of length $q = |c_i|$ with respect to the words $r_1, \ldots, r_m$ (of length $1 < d$), we obtain that R is spanned by all

$$w = u_1 \ldots u_{p_0} v_1^{k_1} u_{p_0+1} \ldots u_{p_1} v_2^{k_2} \ldots v_t^{k_t} u_{p_{t-1}+1} \ldots u_{p_t}.$$

Hence the height of R is bounded by the sum of t and $|c_0| + \ldots + |c_t| = p_t$ and both t and p_t are bounded in terms of d and m, e.g.

$$t + p_t < dm^d + \frac{d^6 m^d}{2}.$$

Exercise 9.2.14 Show that for an m-generated PI-algebra R satisfying a polynomial identity of degree d, the height (with respect to all words of the generators of length $< d$) is bounded by $h \leq \mathcal{O}(d^5 m^d)$.

Hint. Analyse the proof of Shirshov Theorem 9.2.13. Divide each of the words c_i, $i = 0, 1, \ldots, t$, in parts of length $d - 1$ and eventually a shorter residual part. Estimate the total number of subwords obtained as a result of this dividing. It will be $\mathcal{O}(d^5 m^d)$. Add the number $t = \mathcal{O}(dm^d)$ for the words $v_1^{k_1}, \ldots, v_t^{k_t}$.

The theorem of Shirshov may be generalized in two directions. One of them is to find a better bound for the height of the algebra in terms of the number of the generators and the degree of the polynomial identity. The other possibility is to find better bound for the length of the words used in the theorem. From the first point of view, the best known estimate

$$h \leq \frac{(d^5 - d^4) m^d}{2}$$

is obtained from the proof of Belov, a simplified version of which we have presented. Concerning the length of the words involved in the definition of the height of the algebra, it turns out that it is closely related with polynomial identities of matrices.

Definition 9.2.15 The PI-algebra R is of *PI-degree* k, if k is the largest integer such that all multilinear polynomial identities of R follow from the multilinear identities of $M_k(K)$.

Belov [25] showed that, if the algebra satisfies a multilinear polynomial identity which does not hold for the $(k+1)\times(k+1)$ matrix algebra, then the words in the Shirshov theorem can be chosen of length $\leq k$. In an implicit form the result of Belov is contained also in the works of Ufnarovski [252] and Chekanu [48].

Theorem 9.2.16 (Belov-Ufnarovski-Chekanu [25, 252, 48]) *If R is a finitely generated PI-algebra of PI-degree $\leq k$, then R is of finite height with respect to all words of length $\leq k$ of the generators of R.*

Various generalizations of the Shirshov theorem based on combinatorics of words (including infinite words) are given in the survey by Belov, Borisenko and Latyshev [27]. In particular, their article contains different relations between the bounds for the height with respect to the words of length $< d$ and those of length $\leq k$ (where d is the degree of the polynomial identity and k is the PI-degree). Part of these results are contained also in the master's thesis of Asparouhov [15] (in Bulgarian). There one can also find calculations involving the height of concrete relatively free algebras.

Now we apply Shirshov Theorem 9.2.13 to obtain the results of Levitzki and Kaplansky on Kurosch Problem 9.1.4. Pay attention that the statements below do not require the nil property or the algebraicity of all elements of the algebra as in the original results of Levitzki and Kaplansky.

Theorem 9.2.17 *Let R be a PI-algebra satisfying a polynomial identity of degree d and generated by a finite number of elements $r_1,\ldots,r_m$. Let P be the set of all products $r_{i_1}\ldots r_{i_k}$, $k<d$. Then*

(i) (Levitzki [172]) *If every element of the set P is nil, then the algebra R is nilpotent.*

(ii) (Kaplansky [132]) *If every element of P is algebraic, then the algebra R is finite dimensional.*

Proof. By Shirshov Theorem 9.2.13, R is spanned by the products

$$w=u_{i_1}^{k_1}\ldots u_{i_t}^{k_t},$$

where t is bounded by the height h and u_{i_j} are words of length $< d$ in the generators $r_1,\ldots,r_m$. Since all possible words u_{i_j} are a finite number, there is an upper bound n for the class of their nilpotency or for the degree of their algebraicity. Hence, if all u_{i_j} are nil and the sum $k_1+\ldots+k_t$ is sufficiently large (e.g. $> h(n-1)$), then some u_{i_j} appears of degree higher than $n-1$ and the word is equal to 0. If the elements u_{i_j} are algebraic of degree $\leq n$, then the higher degrees of u_{i_j} can be expressed as linear combinations of

$1, u_{i_j}, \ldots, u_{i_j}^{n-1}$. Hence R is spanned by all words w with $k_i < n$ and the number of these words is finite.

9.3 Growth of Algebras and Gelfand-Kirillov Dimension

In this section we shall deal with growth and Gelfand-Kirillov dimension of arbitrary finitely generated algebras. For further reading we recommend the book by Krause and Lenagan [157] and the survey by Ufnarovski [253].

Definition 9.3.1 Let Φ be the set of all functions $f : \mathbb{N} \cup \{0\} \longrightarrow \mathbb{R}$ which are eventually monotone increasing and positive valued. This means that there exists an $n_0 \in \mathbb{N}$ such that $f(n_0) > 0$ and $f(n_2) \geq f(n_1) \geq f(n_0)$ for all $n_2 \geq n_1 \geq n_0$. Define a partial ordering in Φ assuming that $f \preceq g$ for $f, g \in \Phi$ if and only if there exist positive integers a and p such that for all sufficiently large n the inequality $f(n) \leq ag(pn)$ holds and an equivalence assuming that $f \sim g$ for $f, g \in \Phi$ if and only if $f \preceq g$ and $g \preceq f$. We call the equivalence class

$$G(f) = \{g \in \Phi \mid f \sim g\}$$

the *growth* of f.

Exercise 9.3.2 If $f(n)$ and $g(n)$ are polynomial functions with positive coefficients of the leading terms, then $f \preceq g$ if and only if $\deg f \leq \deg g$. If $\alpha, \beta > 0$, then $n^\alpha \sim n^\beta$ if and only if $\alpha = \beta$. The functions α^n and β^n are equivalent if and only if simultaneously either $\alpha = \beta = 1$ or $\alpha, \beta > 1$ ($\alpha^n \notin \Phi$ for $0 < \alpha < 1$).

Definition 9.3.3 Let R be a finitely generated (not obligatorily associative) algebra and let $\{r_1, \ldots, r_m\}$ be a set of generators. Let

$$V^n = \text{span}\{r_{i_1} \ldots r_{i_n} \mid i_j = 1, \ldots, m\},\ n = 0, 1, 2, \ldots,$$

where we assume that $V^0 = K$ if R is unitary and $V^0 = 0$ if it is not unitary (and for nonassociative algebras the parentheses in the products $r_{i_1} \ldots r_{i_n} = (r_{i_1} \ldots)(\ldots r_{i_n})$ are distributed in all possible ways). The *growth function* of R is defined by

$$g_V(n) = \dim(V^0 + V^1 + \ldots + V^n),\ n = 0, 1, 2, \ldots$$

The equivalence class $\mathcal{G}(R) = G(g_V)$ is called the *growth* of R (with respect to the generating vector space V).

Exercise 9.3.4 Show that the growth of a finitely generated algebra R does not depend on the chosen system of generators. If $V = \text{span}\{r_1, \ldots, r_m\}$

and $W = \operatorname{span}\{s_1, \dots, s_{m'}\}$ are the vector spaces spanned by two systems of generators of R, then $G(g_V) = G(g_W)$.

Hint. Since $r_1, \dots, r_m$ generate R, there exists a p such that all s_j are in $V^0 + V^1 + \dots + V^p$. Therefore

$$W^0 + W^1 + \dots + W^n \subseteq V^0 + V^1 + \dots + V^{pn},\ n = 0, 1, 2, \dots,$$

and $g_W(n) \leq g_V(pn)$ for every $n \in \mathbb{N}$. Similarly we obtain that there exists a $q \in \mathbb{N}$ such that $g_V(n) \leq g_W(qn)$, $n \in \mathbb{N}$. Hence $G(g_V) = G(g_W)$.

Definition 9.3.5 Let R be a finitely generated algebra, with a set of generators $\{r_1, \dots, r_m\}$, and let $g_V(n)$ be the growth function of R, where $V = \operatorname{span}\{r_1, \dots, r_m\}$. The *Gelfand-Kirillov dimension of* R is defined by

$$\operatorname{GKdim}(R) = \limsup_{n\to\infty}(\log_n g_V(n)) = \limsup_{n\to\infty} \frac{\log g_V(n)}{\log n}.$$

Exercise 9.3.6 Show that the Gelfand-Kirillov dimension of a finitely generated algebra does not depend on the choice of the set of generators.

Hint. Let V and W be finite dimensional subspaces spanned respectively by two systems of generators of the algebra R and let $\operatorname{GKdim}_V(R)$ and $\operatorname{GKdim}_W(R)$ be the Gelfand-Kirillov dimensions of R defined by means of V and W. By Exercise 9.3.4, there exists a $p \in \mathbb{N}$ such that $g_W(n) \leq g_V(pn)$. Hence

$$\log_n g_W(n) \leq \log_n g_V(pn) = \frac{\log_{pn} g_V(pn)}{\log_{pn} n} = \frac{\log_{pn} g_V(pn)}{1 - \log_{pn} p}$$

and

$$\limsup_{n\to\infty} \log_n g_W(n) \leq \limsup_{pn\to\infty} \frac{\log_{pn} g_V(pn)}{1 - \log_{pn} p} =$$

$$= \limsup_{pn\to\infty} \log_{pn} g_V(pn) \leq \limsup_{n\to\infty} \log_n g_V(n).$$

Hence $\operatorname{GKdim}_W(R) \leq \operatorname{GKdim}_V(R)$. Similarly $\operatorname{GKdim}_V(R) \leq \operatorname{GKdim}_W(R)$.

Exercise 9.3.7 Show that, in the above notation,

$$\operatorname{GKdim}(R) = \inf_\alpha(\alpha \in \mathbb{R} \mid \mathcal{G}(R) \preceq G(n^\alpha)).$$

Hint. If $\operatorname{GKdim}(R) < \alpha$, then for sufficiently large n

$$\log_n g_V(n) < \alpha = \log_n n^\alpha,\ g_V(n) < n^\alpha,\ G(g_V(n)) \preceq G(n^\alpha).$$

Similarly, if $\mathrm{GKdim}(R) > \alpha$, then there exists an $\varepsilon > 0$ and a sequence $n_1 < n_2 < \ldots$ such that

$$n_k^{\alpha+\varepsilon} \leq g_V(n_k),\ k = 1, 2, \ldots$$

This shows that the inequality $G(g_V(n)) \preceq G(n^\alpha)$ is impossible because the function $n_k^{\alpha+\varepsilon}$ grows faster than the function n_k^α.

Exercise 9.3.8 Show that $\mathrm{GKdim}(K[x_1, \ldots, x_m]) = m$.

Hint. The number of all monomials of degree $\leq n$ in m variables is equal to the number of all monomials of degree n in $m+1$ variables since for $a_1 + \ldots + a_m \leq n$ there exists a 1-1 correspondence

$$x_1^{a_1} \ldots x_m^{a_m} \to x_0^{a_0} x_1^{a_1} \ldots x_m^{a_m},\ a_0 = n - (a_1 + \ldots + a_m)$$

between these sets. Hence with respect to the usual set of generators of $R = K[x_1, \ldots, x_m]$,

$$g(n) = \binom{n+m}{m}$$

which is a polynomial of degree m. Hence $\mathcal{G}(R) = G(n^m)$.

Exercise 9.3.9 Show that the free associative algebra $K\langle x_1, \ldots, x_m\rangle$, $m > 1$, has no finite Gelfand-Kirillov dimension.

Hint. For the usual set of generators of $K\langle x_1, \ldots, x_m\rangle$, the growth function is equal to $g(n) = 1 + m + m^2 + \ldots + m^n$ and this function grows faster than any polynomial function.

Exercise 9.3.10 Let R be a finitely generated associative or Lie algebra. Show that $\mathrm{GKdim}(R) = 0$ if and only if R is a finite dimensional vector space. Otherwise $\mathrm{GKdim}(R) \geq 1$.

Hint. Let R be generated by a finite dimensional vector space V. Use that for associative and Lie algebras $V^{n+1} = V^n \cdot V$ and $V^{n+1} = [V^n, V]$, respectively. If $V^{n_0+1} \subseteq V^0 + V^1 + \ldots + V^{n_0}$ for some n_0, then $V^{n+1} \subseteq V^0 + V^1 + \ldots + V^{n_0}$ for all $n \geq n_0$ and R is a finite dimensional vector space. Clearly, in this case $\mathrm{GKdim}(R) = 0$. Otherwise $V^0 + V^1 + \ldots + V^n \neq V^0 + V^1 + \ldots + V^{n+1}$ for all $n = 1, 2, \ldots$ and $g_V(n) \geq n$. Hence $\mathrm{GKdim}(R) \geq 1$.

Exercise 9.3.11 Show that if R is a finitely generated algebra and S is a homomorphic image of R, then $\mathrm{GKdim}(R) \geq \mathrm{GKdim}(S)$.

Hint. Using a set of generators of R and their images as a set of generators of S, we see that $\mathcal{G}(S) \preceq \mathcal{G}(R)$.

Exercise 9.3.12 If R is a finitely generated algebra and S is a finitely generated subalgebra of R, then $\mathrm{GKdim}(R) \geq \mathrm{GKdim}(S)$.

Hint. Let W be the vector space spanned by a set of generators of S. Extend it to a vector space V by a set of generators of R. Then $W^n \subseteq V^n$ and $\mathcal{G}(S) \preceq \mathcal{G}(R)$.

Exercise 9.3.13 Let R be a finitely generated commutative associative algebra and let S be a finitely generated subalgebra of R such that R is a finitely generated S-module. Show that $\mathrm{GKdim}(R) = \mathrm{GKdim}(S)$.

Hint. Let R be generated as an S-module by $r_1, \ldots, r_m$ and let

$$r_i r_j = \sum_{k=1}^{m} s_{ij}^{(k)} r_k,\ s_{ij}^{(k)} \in S.$$

If S is generated by $s_1, \ldots, s_p$, consider $W = \mathrm{span}\{s_{ij}^{(k)}, s_l\} \subseteq S$ and the generating space $V = W + \mathrm{span}\{r_1, \ldots, r_m\} \subseteq R$. Show that

$$\sum_{p=0}^{n} V^p \subseteq \sum_{p=0}^{n} W^p + \sum_{k=1}^{m} \sum_{p=0}^{n-1} W^p r_k.$$

Hence $g_V(n) \leq (m+1) g_W(n)$ and $\mathcal{G}(R) \preceq \mathcal{G}(S)$. On the other hand, Exercise 9.3.12 gives that $\mathcal{G}(S) \preceq \mathcal{G}(R)$. Hence $\mathcal{G}(R) \sim \mathcal{G}(S)$ and $\mathrm{GKdim}(R) = \mathrm{GKdim}(S)$.

Exercise 9.3.14 Let R be a finitely generated commutative associative algebra. Using Noether Normalization Theorem (that R contains a polynomial subalgebra S such that R is a finitely generated S-module, see e.g. [161]), show that the Gelfand-Kirillov dimension of R is equal to the transcendence degree of R (i.e. to the maximal number of algebraically independent elements).

Hint. Use that the Gelfand-Kirillov dimension of S is equal to the number of the free generators of S and apply the previous Exercise 9.3.13.

Exercise 9.3.15 Let G be a finitely generated Lie algebra and let $U = U(G)$ be its universal enveloping algebra. Show that $\mathrm{GKdim}(U) = \dim_K G$ if G is a finite dimensional vector space and U has no finite Gelfand-Kirillov dimension if G is infinite dimensional.

Hint. Let $\dim G = m$ and let G has a basis $g_1, \ldots, g_m$. Then U is generated by the vector space G and Poincaré-Birkhoff-Witt Theorem 1.3.2 gives that

$$G^0 + G^1 + \ldots + G^n = \mathrm{span}\{g_1^{a_1} \ldots g_m^{a_m} \mid a_1 + \ldots + a_m \leq n\}.$$

Hence the growth $\mathcal{G}(U)$ of U is equal to the growth of the polynomial algebra $K[g_1, \ldots, g_m]$ in m commuting variables $g_1, \ldots, g_m$. Let $\dim G = \infty$ and let G be generated by some linearly independent elements $g_1, \ldots, g_k$. Fix a positive integer $m \geq k$ and choose elements $g_{k+1}, \ldots, g_m$ in G such that $g_1, \ldots, g_m$ are linearly independent. Let $V_m = \text{span}\{g_1, \ldots, g_m\}$. Then U is generated by V_m. By Poincaré-Birkhoff-Witt Theorem the monomials $g_1^{a_1} \ldots g_m^{a_m}$ with $a_1 + \ldots + a_m \leq n$ are linearly independent. Therefore,

$$g_V(n) = \dim\left(\sum_{p=0}^{n} V_m^p\right) \geq \binom{n+m}{m}$$

which is a polynomial of degree m. Hence $\text{GKdim}(U) \geq m$ for every $m \geq k$.

9.4 Gelfand-Kirillov Dimension of PI-Algebras

Now we shall prove the theorem of Berele [29] that the finitely generated PI-algebras have finite Gelfand-Kirillov dimension. The original proof of Berele is based on another (weaker) version of Shirshov Theorem 9.2.13. A proof based on structure theory of rings can be found in the book by Krause and Lenagan [157].

Theorem 9.4.1 (Berele [29]) *Every finitely generated PI-algebra has finite Gelfand-Kirillov dimension.*

Proof. Let R be generated by $V = \text{span}\{r_1, \ldots, r_m\}$ and let it satisfy a polynomial identity of degree d. By Shirshov Theorem 9.2.13, there exists an integer h, the height of R, such that R is spanned by the words

$$u_{i_1}^{k_1} \ldots u_{i_t}^{k^t},\ t \leq h,$$

where $u_{i_1}, \ldots, u_{i_t}$ are all words of length $< d$. In the proof of the Shirshov theorem we have used only that R satisfies a multilinear polynomial identity. Hence V^n is spanned on those words with

$$k_1|u_{i_1}| + \ldots + k_t|u_{i_t}| = n.$$

Therefore, $V^0 + V^1 + \ldots + V^n$ is a subspace of the vector space spanned by all words

$$u_{i_1}^{k_1} \ldots u_{i_h}^{k_h},\ k_1 + \ldots + k_h \leq n.$$

Let p be the number of words of length $< d$ ($p = 1 + m + \ldots + m^{d-1}$). The number of sequences of indices $(i_1, \ldots, i_h)$ is bounded by p^h. Hence the dimension $g_V(n)$ of $V^0 + V^1 + \ldots + V^n$ is bounded by the product of the number of sequences $(i_1, \ldots, i_h)$ and the number of monomials of degree $\leq n$ in h variables,

$$g_V(n) \leq p^h \binom{n+h}{h}$$

which is a polynomial of degree h. Hence

$$\mathrm{GKdim}(R) \leq h,$$

the height of R.

We state as a corollary the last inequality.

Corollary 9.4.2 *The Gelfand-Kirillov dimension of a finitely generated PI-algebra R is bounded by the height of R with respect to any finite set of monomials of any finite set of generators of R.*

Exercise 9.4.3 If R is an m-generated PI-algebra with a polynomial identity of degree d, show that $\mathrm{GKdim}(R) < dm^d$.

Hint. Use Theorem 9.2.9 and modify the proof of Theorem 9.4.1. Show that

$$R = \mathrm{span}\{c_0 v_1^{k_1} c_1 \dots v_r^{k_r} c_r \mid \sum_{i=0}^{r} |c_i| < a,\ k_1, \dots, k_r \geq 0,\ r < dm^d\}$$

for a being fixed implies that $\mathrm{GKdim}(R) < dm^d$.

Exercise 9.4.4 Let $\mathrm{char} K = 0$ and let $R = F_m(E)$ be the relatively free algebra in the variety generated by the Grassmann algebra. Show that $\mathrm{GKdim}(R) = m$.

Hint. Use Theorem 4.3.11 (i) and the proof of Theorem 5.1.2. As a graded vector space R is a direct sum of a finite number of copies of

(the polynomial algebra) × (homogeneous element)

because R has a basis

$$x_1^{a_1} \dots x_m^{a_m} [x_{i_1}, x_{i_2}] \dots [x_{i_{2k-1}}, x_{i_{2k}}],\ 1 \leq i_1 < i_2 < \dots < i_{2k-1} < i_{2k} \leq m.$$

Starting with the set of generators $\{x_1, \dots, x_m\}$, we obtain

$$\mathcal{G}(R) = \mathcal{G}(K[x_1, \dots, x_m]).$$

Exercise 9.4.5 Let the field K be infinite and let $R = F_m(U_k(K))$ be the relatively free algebra of the variety generated by the algebra of $k \times k$ upper triangular matrices. Show that $\mathrm{GKdim}(R) = mk$.

Hint. Use the idea in the previous Exercise 9.4.4. As in Exercise 9.2.12, R is spanned by

$$x_1^{a_1}\dots x_m^{a_m}[x_{i_1},x_{j_1}]x_1^{b_1}\dots x_m^{b_m}[x_{i_2},x_{j_2}]\dots[x_{i_p},x_{j_p}]x_1^{c_1}\dots x_m^{c_m},\ p<k,$$

and the growth of R is bounded from above by the growth of a finite direct sum of polynomial algebras in mc variables. On the other hand, show that the following elements are linearly independent in R which will give that the growth of R is bounded from below by the growth of the same polynomial algebra in mc variables:

$$x_1^{a_1}\dots x_m^{a_m}[x_1,x_2]x_1^{b_1}\dots x_m^{b_m}[x_1,x_2]\dots[x_1,x_2]x_1^{c_1}\dots x_m^{c_m},$$

where the number of the commutators is $c-1$.

We give a short survey on some results concerning growth and Gelfand-Kirillov dimension of relatively free algebras. For more detailed exposition see [87]. Although some of the results below hold for any field or for any infinite field, we assume that char$K=0$. The Gelfand-Kirillov dimension of the algebra generated by m generic $k\times k$ matrices is equal to the transcendence degree of its centre and is $(m-1)k^2+1$ (see the book by Procesi [213]). The asymptotic behaviour of the growth of some relatively free algebras was studied in the paper of Grishin [120]. In his survey article [106] Formanek gave a formula for the Hilbert series of the product of two T-ideals as a function of the Hilbert series of the factors (see Halpin [125] for the proof). A translation of the formula of Formanek in the language of relatively free algebras and codimensions can be found for example in the paper by Drensky [76]. Using the result of Formanek, Berele [30, 31] calculated the Gelfand-Kirillov dimension of some relatively free algebras with T-ideals which are products of T-prime T-ideals (in the classification of Kemer, see Theorem 8.4.10). Markov [183] announced that for relatively free algebras $F_m(R)$ satisfying a nonmatrix polynomial identity, the Gelfand-Kirillov dimension is the same as the Gelfand-Kirillov dimension of the algebra $F_m(U_k(K))$, where $U_k(K)$ is the largest algebra of upper triangular matrices satisfying all polynomial identities of R. Theorem 6.2.5 of Belov [26] for the rationality of the Hilbert series of relatively free algebras, combined with Theorem 9.4.1 of Berele, implies that GKdim$(F_m(R))$ is an integer for any PI-algebra R and any m. This result follows also from the theorem of Kemer (see [142]) that the relatively free algebra $F_m(R)$ is *representable* (i.e. isomorphic to a subalgebra of the K-algebra $M_k(S)$ for some k and some commutative algebra S) and the theorem of Markov (announced in [183]) that the Gelfand-Kirillov dimension of a finitely generated representable algebra is an integer.

Since Shirshov Theorem 9.2.13 gives a bound for the Gelfand-Kirillov dimension, it is interesting to see how far is this bound from the real value of the Gelfand-Kirillov dimension. Since the Shirshov theorem uses only the existence of a polynomial identity, the experiment has to be made correctly, i.e. for relatively free algebras. For example, comparing Exercises 9.2.12, 9.2.14, 9.4.3 and 9.4.5 (and their hints) we see that for $F_m(U_k(K))$ the Gelfand-Kirillov dimension can be obtained from the Shirshov theorem. It turns out that one can define the so called *essential height* which is a modification of the original notion of the height introduced by Shirshov (see the survey article of Belov, Borisenko and Latyshev [27] and the master's thesis of Asparouhov [15]). In particular, the Gelfand-Kirillov dimension of a finitely generated presentable algebra coincides with its essential height [27]. Calculations with the height and the Gelfand-Kirillov dimension of concrete relatively free algebras are given in [15].

The Gelfand-Kirillov dimension of an arbitrary finitely generated commutative algebra is an integer. We shall give a modification of the example of Borho and Kraft [36], showing that the Gelfand-Kirillov dimension of a finitely generated PI-algebra can be equal to any real $\alpha \geq 2$. There are also some other examples in the book by Krause and Lenagan [157] and the survey by Ufnarovski [253]. A result of Bergman (see [157]) shows that there exist no algebras with Gelfand-Kirillov dimension in the interval $(1,2)$. Hence the Gelfand-Kirillov dimension of a finitely generated associative algebra can have as a value 0, 1 and any $\alpha \geq 2$. We also give an example of a finitely generated Lie algebra with polynomial identity and with Gelfand-Kirillov dimension any α in the interval $(1,2)$. This is a special case of the examples given by Petrogradsky [206] and showing that there exist Lie algebras with any prescribed Gelfand-Kirillov dimension $\alpha \geq 1$.

Lemma 9.4.6 *Let $f(u)$ and $g(u)$ be two continuous monotone increasing functions defined for every $u \geq 0$. Let*

$$a(t) = a_1t + a_2t^2 + \ldots,\ b(t) = \frac{a(t)}{1-t} = b_1t + b_2t^2 + \ldots$$

be formal power series such that

$$f(n) \leq a_n \leq g(n),\ n = 1,2,\ldots$$

Then

$$\int_0^n f(u)du \leq b_n = a_1 + a_2 + \ldots + a_n \leq \int_1^{n+1} g(u)du.$$

Proof. For every $k = 1,2,\ldots,n$,

$$f(k-1) \leq f(u) \leq f(k) \leq a_k,\ k-1 \leq u \leq k+1.$$

Hence

$$\int_{k-1}^k f(u)du \leq a_k,\ \int_0^n f(u)du = \sum_{k=1}^n \int_{k-1}^k f(u)du \leq a_1 + a_2 + \ldots + a_n = b_n.$$

The other inequality can be obtained analogously.

Till the end of the section we fix an integer $k \geq 2$ and the algebra R generated by two elements x and y with defining relations

$$yx^{p_1}yx^{p_2}y\dots yx^{p_k}y=0,\ p_i\geq 0,\ i=1,\dots,k.$$

Hence all words containing $k+1$ times the variable y are equal to 0 in R. We assume that the base field K is arbitrary and the algebra R is graded in the canonical way.

Lemma 9.4.7 *The algebra R has a basis*

$$\{x^{p_1}yx^{p_2}y\dots x^{p_s}yx^{p_{s+1}}\mid p_i\geq 0, s\leq k\}.$$

The Gelfand-Kirillov dimension of R is equal to $k+1$.

Proof. The free algebra $K\langle x,y\rangle$ has a basis

$$\{x^{p_1}yx^{p_2}y\dots x^{p_s}yx^{p_{s+1}}\mid p_i\geq 0, s=0,1,2\dots\},$$

and the vector space I spanned by the products with $s>k$ is an ideal of $K\langle x,y\rangle$. Hence $R\cong K\langle x,y\rangle/I$ and the first part of the lemma is established. Therefore the Hilbert series of R is equal (why?) to

$$\mathrm{Hilb}(R,t)=\sum_{n\geq 0}\dim R^{(n)}t^n=\frac{1}{1-t}+t\frac{1}{(1-t)^2}+\dots+\frac{t^k}{(1-t)^{k+1}}.$$

The coefficient $h_n=\dim R^{(n)}$ is a polynomial of degree k in n. Hence $h_0+h_1+\dots+h_n$ is a polynomial of degree $k+1$ (why?) and this shows that $\mathrm{GKdim}(R)=k+1$.

Exercise 9.4.8 (i) Show that the algebra R satisfies the polynomial identity

$$[x_1,x_2]\dots[x_{2k+1},x_{2k+2}]=0.$$

(ii) Show that R has the following matrix presentation

$$x=\xi_1e_{11}+\xi_2e_{22}+\dots+\xi_{k+1}e_{k+1,k+1},\ y=e_{12}+e_{23}+\dots+e_{k,k+1},$$

where $\xi_1,\dots,\xi_{k+1}$ are independent commuting variables.

Hint. (i) Use that every nonzero commutator of monomials in R is a difference of two monomials, each containing the variable y.

(ii) Show that the algebra generated by these two matrices satisfies the relations of R and the basis elements of R are linearly independent in the considered matrix algebra. (Repeat the arguments in the proof of Theorem 5.2.1 for the basis of the relatively free algebra in the variety generated by the algebra of upper triangular matrices.)

For any real number $0<\beta<1$ we build a sequence $a_0,a_1,\dots$ in the following way: $a_0=0$, $a_1=1$, and, inductively,

$$a_n = 0, \text{ if } a_1 + a_2 + \ldots + a_{n-1} = [n^\beta],$$

$$a_n = 1, \text{ if } a_1 + a_2 + \ldots + a_{n-1} < [n^\beta],$$

where $[n^\beta]$ is the integer part of n^β. In this way, for every $n = 1, 2, \ldots$,

$$n^\beta - 1 < a_1 + a_2 + \ldots + a_n = [n^\beta] \leq n^\beta.$$

Exercise 9.4.9 Show that for any $\beta \in (0, 1)$ it is possible to construct the above sequence $a_0, a_1, a_2, \ldots$

Hint. Use calculus. By the mean value theorem, for every $u \geq 1$

$$\frac{(u+1)^\beta - u^\beta}{(u+1) - u} = \beta\xi^{\beta-1} < 1,$$

for some $\xi \in (u, u+1)$. Hence the difference between n^β and $(n+1)^\beta$ is less than 1 and this means that $[(n+1)^\beta] - [n^\beta] \leq 1$. Choosing $a_{n+1} = 0$ or $a_{n+1} = 1$ we always may make the sum $a_1 + \ldots + a_{n+1}$ equal to $[(n+1)^\beta]$.

Now we consider the vector subspace J_β of R spanned by all products

$$x^{p_1}yx^{p_2}y \ldots x^{p_k}yx^{p_{k+1}},$$

where $p_1, p_3, p_4, \ldots, p_{k+1}$ are arbitrary nonnegative integers and p_2 runs on the set of all n with $a_n = 0$. Clearly, J_β is an ideal of R because $yJ_\beta = J_\beta y = 0$ and the multiplication by x preserves the property for p_2.

For example, for $\beta = 0.5$, by definition $a_0 = 0$, $a_1 = 1$, and

$$[\sqrt{2}] = [\sqrt{3}] = 1,\ a_2 = a_3 = 0,$$

$$[\sqrt{4}] = 2 > [\sqrt{3}],\ a_4 = 1, \ldots$$

For $k = 2$, the ideal $J_{0.5}$ contains all

$$x^{p_1}yx^0yx^{p_3},\ x^{p_1}yx^2yx^{p_3},\ x^{p_1}yx^3yx^{p_3}$$

because $a_0 = a_2 = a_3 = 0$, and does not contain

$$x^{p_1}yx^1yx^{p_3},\ x^{p_1}yx^4yx^{p_3}$$

because $a_1 = a_4 = 1$.

Lemma 9.4.10 *The Hilbert series of the factor algebra R/J_β is equal to*

$$\mathrm{Hilb}(R/J_\beta, t) = \sum_{q=0}^{k-1} \frac{t^q}{(1-t)^{q+1}} + \frac{a(t)t^k}{(1-t)^k},$$

where $a(t) = t + a_2t^2 + a_3t^3 + \ldots$ is the generating function of the sequence $a_0, a_1, a_2, \ldots$ defined above.

Proof. The proof follows easily by counting the basis elements of R/J_β. The nominators t^q indicate the multiplicity q of the appearance of y in the monomials of the basis of R/J_β. The denominators $(1-t)^{q+1}$ stay because for monomials containing q times y, the x's behave as the commuting variables $x_1, \ldots, x_{q+1}$ (x^p between i-th and $i+1$-st y's is the same as x_{i+1}^p). Finally, in the monomials with k y's, the power x^p appears between the first and second y's if and only if $a_p = 1$.

Theorem 9.4.11 *For any real $\alpha \geq 2$ there exists an algebra with two generators and with Gelfand-Kirillov dimension equal to α.*

Proof. If α is an integer, then it is easy to find an algebra with Gelfand-Kirillov dimension equal to α. For $\alpha = 2$ we can consider the polynomial algebra $K[x,y]$ in two commuting variables. For $\alpha \geq 3$ the algebra R defined above for $k = \alpha - 1$ has Gelfand-Kirillov dimension $k+1 = \alpha$. Now let $k < \alpha < k+1$. We choose $\beta = \alpha - k$. We shall show that $\mathrm{GKdim}(R/J_\beta) = \alpha$ for the algebra R/J_β defined in Lemma 9.4.10. By the same lemma, the vector space R/J_β is graded and its Hilbert series is equal to

$$\mathrm{Hilb}(R/J_\beta, t) = \sum_{n\geq 0} b_n t^n = \sum_{q=0}^{k-1} \frac{t^q}{(1-t)^{q+1}} + \frac{a(t)t^k}{(1-t)^k},$$

where $a(t) = a_1 t + a_2 t^2 + a_3 t^3 + \ldots$ satisfies the condition

$$n^\beta - 1 < a_1 + \ldots + a_n = [n^\beta] \leq n^\beta.$$

Hence, by Lemma 9.4.6, the coefficients of the series

$$\sum_{n\geq 1} c_n^{(p)} t^n = \frac{a(t)}{(1-t)^p}$$

satisfy the inequalities

$$\frac{1}{\beta+1} n^{\beta+1} - n = \int_0^n (u^\beta - 1)du \leq c_1^{(1)} + c_2^{(1)} + \ldots + c_n^{(1)} \leq$$

$$\leq \int_1^{n+1} u^\beta du = \frac{1}{\beta+1}((n+1)^{\beta+1} - 1),$$

and, by induction on p,

$$\gamma n^{k+\beta} - f_1(n) \leq c_1^{(k)} + c_2^{(k)} + \ldots + c_n^{(k)} \leq \gamma(n+k)^{k+\beta} + f_2(n),$$

for some positive $\gamma \in \mathbb{R}$ and some polynomials $f_1(x), f_2(x) \in \mathbb{R}[x]$ of degree $\leq k$. Since for n sufficiently large, the coefficients of the part of the Hilbert series of R/J_β

$$\sum_{q=0}^{k-1}\frac{t^q}{(1-t)^{q+1}}$$

are also polynomials of degree $k-1$, we obtain that

$$\gamma n^{k+\beta}-h_1(n)\leq b_0+b_1+\ldots+b_n\leq \gamma n^{k+\beta}+h_2(n)$$

for $0<\gamma\in\mathbb{R}$, $h_1(x),h_2(x)\in\mathbb{R}[x]$ and $\deg h_i\leq k$, $i=1,2$. This implies that

$$\mathcal{G}(R/J_\beta)=G(n^{k+\beta})=G(n^\alpha),$$

where $G(n^\alpha)$ is the equivalence class of n^α. Hence $\mathrm{GKdim}(R/J_\beta)=\alpha$.

Exercise 9.4.12 (Petrogradsky [206]) For any $\alpha\in(1,2)$, there exists a Lie algebra G_α with two generators and with Gelfand-Kirillov dimension equal to α.

Hint. Consider the associative algebra R generated by x and y and with defining relations

$$yyx^ay=0,\ xyx^ay=0,\ a\geq 0.$$

Let $G=R^{(-)}$ be the vector space R, considered as a Lie algebra with multiplication $[u,v]=uv-vu$ (as in Exercise 1.1.10 (ii)). Show that G has a basis consisting of x,y,

$$y\mathrm{ad}^kx=[y,x,x,\ldots,x],[y\mathrm{ad}^kx,y]=[y,x,x,\ldots,x,y],\ k=1,2,\ldots,$$

$$u_{pq}=[y\mathrm{ad}^px,y\mathrm{ad}^qx]=[[y,x,x,\ldots,x],[y,x,\ldots,x]],\ p>q\geq 1.$$

Show that the elements u_{pq} span the ideal G'' of G. Calculate the Hilbert series of G

$$\mathrm{Hilb}(G,t)=2(t+t^2+t^3+t^4+\ldots)-t^2+\sum_{n\geq 5}a_nt^n,$$

where a_n is the dimension of the homogeneous component of degree n of G''. The coefficients a_n grow linearly and, hence, faster than $n^{\alpha-1}$. Define inductively in the following way an ideal J of G, such that $J\subseteq G''$. The first elements of J are for the smallest n with $a_5+\ldots+a_n>n^\alpha$. Choose k_n linearly independent elements $v_1,\ldots,v_{k_n}$ of degree n in G'' in such a way that

$$a_5+\ldots+a_n-k_n=[n^\alpha].$$

Let J_n be the ideal generated by $W_n=\mathrm{span}\{v_1,\ldots,v_{k_n}\}$. Consider the subspace $V_{n+1}=[W_n,x]$. Its dimension is $\leq k_n$. Hence

$$a_5+\ldots+a_{n+1}-\dim W_n-\dim V_{n+1}\geq[(n+1)^\alpha].$$

Choose a homogeneous vector space W_{n+1} of degree $n+1$ such that $V_{n+1}\subseteq W_{n+1}\subset G''$ and

$$a_5 + \ldots + a_{n+1} - \dim W_n - \dim W_{n+1} = [(n+1)^\alpha].$$

Generate by $W_n + W_{n+1}$ an ideal J_{n+1} of G. Continuing in this way, we obtain a sequence of ideals $J_n \subseteq J_{n+1} \subseteq \ldots$ such that the coefficients of the Hilbert series $\sum_{n\geq 5} c_n t^n$ of G''/J_p satisfy

$$c_5 + \ldots + c_p = [p^\alpha].$$

Let $J = J_n + J_{n+1} + \ldots$ Derive from here that the Gelfand-Kirillov dimension of the Lie algebra G/J is equal to α.

By Exercise 9.4.8, the example in Theorem 9.4.11 is a homomorphic image of the relatively free algebra $F_2(U_{k+1}(K))$ where $U_{k+1}(K)$ is the associative algebra of $(k+1) \times (k+1)$ upper triangular matrices. Similarly, the example in Exercise 9.4.12 is an image of the relatively free Lie algebra $F_2(U_3^{(-)}(K))$. Drensky and Kassabov (see the master's thesis of Kassabov [136]) showed that for $k \geq 3$, $m \geq 2$ and for any real α in the interval $[2, \mathrm{GKdim}(F_m(U_k(K)))]$ there exists a homomorphic image R of $F_m(U_k(K))$ such that $\mathrm{GKdim}(R) = \alpha$. Similarly, if $1 \leq \alpha \leq \mathrm{GKdim}(F_m(U_k^{(-)}(K)))]$, $k \geq 3$, $m \geq 2$, there exists an image G of the relatively free Lie algebra $F_m(U_k^{(-)}(K))$ with $\mathrm{GKdim}(G) = \alpha$.

10. Automorphisms of Free Algebras

In this chapter we study the automorphism group of the polynomial algebra in m variables and of the free associative algebra of rank m, where m is a fixed positive integer. In particular, we prove the theorem of Jung [130] and Van der Kulk [257] stating that the automorphisms of $K[x,y]$ are tame and its noncommutative analogue, the theorem of Makar-Limanov [177] and Czerniakiewicz [53] for the tameness of the automorphisms of $K\langle x,y\rangle$. The proof of the Jung-Van der Kulk theorem which we give here uses locally nilpotent derivations. We consider also automorphisms of the polynomial algebra in m variables which are exponents of locally nilpotent derivations and show that important automorphisms can be obtained in this way. Finally we extend our considerations to automorphisms of relatively free algebras $F_m(R)$, where R is some PI-algebra. Most of the main results hold (maybe with different proofs) for algebras over any field K and, after some restatement, for nonunitary algebras. Nevertheless we assume that all algebras are unitary and over an algebraically closed field K of characteristic 0.

10.1 Automorphisms of Groups and Algebras

There are two main reasons to study the automorphisms of $K[x_1,\ldots,x_m]$ and $K\langle x_1,\ldots,x_m\rangle$. The first is that the polynomial algebra and the free associative algebra are among the important algebras and the second reason is the following.

Let R be any associative algebra finitely generated by a set $\{r_1,\ldots,r_m\}$. The map

$$x_i \longrightarrow r_i,\ i=1,\ldots,m,$$

can be uniquely extended to a homomorphism

$$\pi : K\langle x_1,\ldots,x_m\rangle \longrightarrow R$$

and

$$R \cong K\langle x_1,\ldots,x_m\rangle/\mathrm{Ker}\,\pi.$$

A possible approach to describe the automorphism group $\mathrm{Aut}R$ of the algebra R is to do this in three steps.

Step 1. Describe the automorphisms of $K\langle x_1,\ldots,x_m\rangle$.
Step 2. Find the automorphisms ψ of $K\langle x_1,\ldots,x_m\rangle$ which *induce* automorphisms of R, i.e. such that $\psi(\mathrm{Ker}\pi)=\mathrm{Ker}\pi$.
Step 3. Try to understand the automorphisms of R which cannot be obtained by the first two steps, i.e. cannot *be lifted* to automorphisms of the free algebra.

Instead of the homomorphism $\pi : K\langle x_1,\ldots,x_m\rangle \longrightarrow R$, we can consider the homomorphism

$$\pi_S : F_m(S) = K\langle x_1,\ldots,x_m\rangle/(K\langle x_1,\ldots,x_m\rangle \cap T(S)) \longrightarrow R,$$

where S is a PI-algebra and R satisfies all polynomial identities of S. It turns out that for many algebras S, the relatively free algebra $F_m(S)$ has "more" automorphisms than the free algebra $K\langle x_1,\ldots,x_m\rangle$ and hence induces more automorphisms for R. This also allows to use powerful techniques typical for the theory of PI-algebras. Since the class of PI-algebras is close to the class of commutative algebras, some noncommutative results have purely commutative consequences. Especially interesting is the algebra of two generic 2×2 matrices, in particular because every of its automorphisms induces an automorphism of the polynomial algebra in five variables (see Remark 10.5.3 below).

A similar approach is possible also for other classes of algebraic systems, e.g. for commutative algebras, Lie algebras, groups, etc. Of course, then we should replace the free associative algebra $K\langle x_1,\ldots,x_m\rangle$ respectively with the polynomial algebra $K[x_1,\ldots,x_m]$, the free Lie algebra of rank m, the free group of rank m, etc. For further reading we recommend the books by Cohn [51] and Kharchenko [147] as well as the collection of papers [102] edited by van der Essen and the two survey articles by Drensky [81, 86].

Since we consider relatively free algebras only, every map $\{x_1,\ldots,x_m\}\to F_m(R)$ can be uniquely extended to an endomorphism of the algebra. Sometimes we shall use the following notation which also appears in the literature and especially in commutative algebra. If $\phi\in \mathrm{End}F_m(R)$, and $\phi(x_i)=f_i$, $i=1,\ldots,m$, we denote ϕ also as

$$\phi=(f_1,\ldots,f_m).$$

We always assume that the endomorphisms act from the left, i.e.

$$(\phi\psi)(u)=(\phi\circ\psi)(u)=\phi(\psi(u)),\ \phi,\psi\in \mathrm{End}F_m(R),\ u\in F_m(R).$$

In order to simplify the notation we fix an m-dimensional vector space V_m with basis $\{x_1,\ldots,x_m\}$ over the field K and denote by $K[V_m]$ and $K\langle V_m\rangle$ respectively the polynomial algebra and the free associative algebra freely generated by $x_1,\ldots,x_m$. For small m we shall also use other letters, e.g. $x,y,\ldots,z$, for the free generators of $K[V_m]$, $K\langle V_m\rangle$ and $F_m(R)$.

One of the main problems under discussion is the following.

Problem 10.1.1 Find a natural and "nice looking" set of generators of the automorphism group of $K[V_m]$, $K\langle V_m\rangle$ and $F_m(R)$, where R is some PI-algebra.

One of the first results in this spirit is for free groups and is due to Nielsen [193].

Theorem 10.1.2 (Nielsen [193]) *The automorphism group of the free group* G_m *of rank* $m \geq 2$ *is generated by the following automorphisms* $\sigma, \theta_1, \theta_2$ *defined by*

(i) $\sigma(x_i) = x_{\sigma_0(i)}$, *where* σ_0 *belongs to the symmetric group* S_m *of degree* m;

(ii) $\theta_1(x_1) = x_1^{-1}$, $\theta_1(x_i) = x_i$, $i \neq 1$;

(iii) $\theta_2(x_1) = x_1x_2$, $\theta_2(x_i) = x_i$, $i \neq 1$.

Traditionally the automorphisms of a finitely generated relatively free group G induced by automorphisms of the free group are called *tame* and the other automorphisms are *wild*. Below we translate the notion of tameness in the language of ring theory.

Exercise 10.1.3 Show that the automorphism group $\mathrm{Aut}K[x]$ of the polynomial algebra in one variable is isomorphic to the *affine group of the line*, and a map $\phi : x \to K[x]$ induces an automorphism of $K[x]$ if and only if $\phi(x) = \alpha x + \beta$, $\alpha, \beta \in K$, $\alpha \neq 0$. Find the inverse of this automorphism.

Hint. Show that $\phi(x) = \alpha x+\beta$ has an inverse of the form $\phi^{-1}(x) = \alpha^{-1}x+\gamma$, $\gamma \in K$. If ϕ is any automorphism of $K[x]$, show that for $h(x) \in K[x]$,

$$\mathrm{deg}\phi(h(x)) = \mathrm{deg}\phi(x)\mathrm{deg}h(x).$$

If $\mathrm{deg}\phi(x) > 1$, then x is not an image of any $h(x) \in K[x]$.

Since the description of the automorphisms of $K[x] = K\langle x\rangle$ is trivial, in this chapter we assume that the integer m is bigger than 1.

Exercise 10.1.4 (i) Let $g = (\alpha_{pq})$ be an invertible $m \times m$ matrix with entries from K and let $\beta_1, \ldots, \beta_m \in K$. Show that the map

$$x_i \to \beta_i + \sum_{p=1}^{m} \alpha_{pi}x_p,\ i = 1, \ldots, m,$$

defines an automorphism of $K[V_m]$.

(ii) Show that the following endomorphism of $K[V_m]$ is an automorphism

$$\phi = (\alpha_1x_1 + f_1, \ldots, \alpha_mx_m + f_m),$$

where $0 \neq \alpha_i \in K$, and the polynomials $f_i = f_i(x_{i+1}, \ldots, x_m)$, $i = 1, \ldots, m$, do not depend on $x_1, \ldots, x_i$ (in particular, f_m is a constant). How to find the inverse of ϕ? Find the inverse of the automorphism of $K[x, y, z]$, where

$$\phi = (2x + y^2 + z, -y + z^2, 3z).$$

(iii) Show that the statements of (i) and (ii) hold also for the free associative algebra and any relatively free algebra.

Hint. (ii) Apply induction on m. For $m = 1$ apply Exercise 10.1.3. For $m > 1$, use that ϕ is invertible on $K[x_2, \ldots, x_m]$ and show that

$$\phi^{-1}(x_1) = \alpha_1^{-1} x_1 - \alpha^{-1} f_1(\phi^{-1}(x_2), \ldots, \phi^{-1}(x_m)).$$

Definition 10.1.5 (i) An automorphism ϕ of the polynomial algebra $K[V_m]$ is called *affine* if

$$\phi(x_i) = \beta_i + \sum_{p=1}^{m} \alpha_{pi} x_p, \ i = 1, \ldots, m,$$

where $g = (\alpha_{pq})$ is an invertible $m \times m$ matrix with entries from K and $\beta_1, \ldots, \beta_m \in K$. If all β_i are equal to 0, then ϕ is called *linear* and in this way we identify the general linear group $GL_m(K)$ with a subgroup of $\mathrm{Aut}K[V_m]$.

(ii) An automorphism ϕ of $K[V_m]$ is *triangular* (or *de Jonquières*), if

$$\phi(x_i) = \alpha_i x_i + f_i(x_{i+1}, \ldots, x_m), \ 0 \neq \alpha_i \in K, \ i = 1, \ldots, m,$$

where the polynomial $f_i(x_{i+1}, \ldots, x_m)$ does not depend on $x_1, \ldots, x_i$.

(iii) An automorphism of $K[V_m]$ is called *tame* if it belongs to the subgroup of $\mathrm{Aut}K[V_m]$ generated by the affine and the triangular automorphisms.

The other automorphisms of $K[V_m]$ (if they exist) are called *wild.*

The notion of tame and wild automorphisms is defined in a similar way for $K\langle V_m \rangle$ and is also extended to the automorphisms of relatively free algebras $F_m(R)$.

The classical form of Problem 10.1.1 is the following.

Problem 10.1.6 Are all automorphisms of $K[V_m]$, $K\langle V_m \rangle$ and $F_m(R)$ tame?

It turns out that this is a very difficult problem. For polynomial algebras and free associative algebras the affirmative answer is known for $m = 2$ only. For $m > 2$ the problem is still open and there are some evidences that probably the answer is negative.

Exercise 10.1.7 Let $\phi = (f_1, \ldots, f_m)$ be an endomorphism of $K[V_m]$ such that

$$f_i = f_i(x_1, \ldots, x_m) = \beta_i + g_i(x_1, \ldots, x_m),\ \beta_i \in K,\ i = 1, \ldots, m,$$

and the polynomials g_i have no constant terms. Show that ϕ is an automorphism if and only if $\psi = (g_1, \ldots, g_m)$ is an automorphism.

Hint. Consider the *translation* $\tau = (x_1 + \beta_1, \ldots, x_m + \beta_m)$. Show that $\phi = \psi \circ \tau$. Since $\tau \in \mathrm{Aut}K[V_m]$, conclude that ϕ is an automorphism if and only if ψ is an automorphism.

Definition 10.1.8 We say that the endomorphism $\psi = (g_1, \ldots, g_m)$ of $K[V_m]$ (or of $K\langle V_m\rangle$, or of $F_m(R)$) *preserves the augmentation* (or is an *endomorphism without constant terms*) if the polynomials $g_i = g_i(x_1, \ldots, x_m)$ have no constant terms. (These endomorphisms have the property that they preserve the *augmentation ideal* of $K[V_m]$, i.e. if h is a polynomial without constant term, then the same is $\psi(h)$.)

The next exercise shows that the existence of unity is not very important in our considerations and we can define tame automorphisms also for nonunitary algebras.

Exercise 10.1.9 Let

$$\phi = (\beta_1 + g_1, \ldots, \beta_m + g_m),\ \psi = (g_1, \ldots, g_m),\ \beta_i \in K,\ i = 1, \ldots, m,$$

and let the polynomials g_i have no constant terms. Show that ϕ is tame if and only if ψ is a product of linear and triangular automorphisms without constant terms.

Hint. Apply Exercise 10.1.7. If ψ is a product of affine and triangular automorphisms without constant terms, use that the translation $\tau : x_i \to x_i + \beta_i$ is a tame automorphism and derive that ϕ is tame. If ϕ_1 is an affine or a triangular automorphism and τ_1 is a translation, show that there exist a translation τ_2 and a linear or a triangular automorphism without constant terms ψ_1 such that $\tau \circ \phi_1 = \psi_1 \circ \tau_2$. Hence, if ϕ is tame, then we can decompose it as $\phi = \psi \circ \tau = \psi_1 \circ \ldots \circ \psi_k \circ \tau$, where each ψ_i is a linear or a triangular automorphism without constant terms and τ is a translation.

Exercise 10.1.10 Let G be the subgroup of all augmentation preserving automorphisms of $K[V_m]$ and let H be the subgroup of the automorphisms

$$\psi : x_i \to x_i + \text{ terms of higher degree},\ i = 1, \ldots, m.$$

Show that H is a normal subgroup of G and G is a product of $GL_m(K)$ and H. The automorphisms of H are called sometimes *IL-automorphisms* (IL = Identical Linear component).

Hint. If

$$\psi : x_i \to \sum_{p=1}^{m} \alpha_{pi} x_p + \text{ terms of higher degree, } i = 1, \dots, m,$$

$\rho = (\alpha_{pq}) \in GL_m(K)$, show that $\rho^{-1} \circ \psi \circ \rho \in H$.

Exercise 10.1.11 Show that an endomorphism ϕ of $K[V_m]$ (or of $K\langle V_m \rangle$) is an automorphism if and only if ϕ is surjective, i.e. $\phi(K[V_m]) = K[V_m]$ (or, respectively, $\phi(K\langle V_m \rangle) = K\langle V_m \rangle$).

Hint. Let ω be the augmentation ideal of $K[V_m]$ (the ideal of polynomials without constant terms) and assume that ϕ is an augmentation preserving endomorphism. Show that ϕ induces a vector space endomorphism ϕ_k of the factors ω^k/ω^{k+1}. Since ϕ is an epimorphism and $\dim \omega^k/\omega^{k+1} < \infty$, we obtain that ϕ_k is an automorphism of ω^k/ω^{k+1}. If h is in the kernel of ϕ and $h \in \omega^k \setminus \omega^{k+1}$, then $\phi_k(h + \omega^{k+1}) = 0$, which is impossible. If ϕ is an arbitrary epimorphism, apply Exercise 10.1.7 to reduce the considerations to the case when ϕ is augmentation preserving.

10.2 The Polynomial Algebra in Two Variables

The main result of this section is the theorem of Jung-Van der Kulk that all automorphisms of the polynomial algebra in two variables are tame. The theorem was proved by Jung [130] for $K = \mathbb{C}$ in 1942 and by Van der Kulk [257] for any field in 1953. There are several other proofs based on different geometric or combinatorial ideas as those of Engel [100], Gutwirth [122], Nagata [190], Abhyankar and Moh [3], etc. See also the paper by Dicks [56], the Ph.D. thesis of Makar-Limanov [178] and the book by Cohn [51]. Some of these proofs use essentially that $\mathrm{char} K = 0$ and other work in the general case. Here we give the recent proof of Makar-Limanov [179], which is quite transparent and holds for algebraically closed fields of characteristic 0. With minor additional efforts his considerations can be modified for any field of characteristic 0. Since the proof of Makar-Limanov uses locally nilpotent derivations of $K[x, y]$, we start the section with some preliminaries on derivations. Recall (see Exercises 2.1.15 and 3.2.1) that a linear mapping ∂ of a (not necessarily commutative or associative) algebra G is a derivation if $\partial(uv) = \partial(u)v + u\partial(v)$ for all $u, v \in G$. As usually, we assume that R is any PI-algebra and $F_m(R)$ is the corresponding relatively free algebra of rank m.

Exercise 10.2.1 Show that every map $\partial : X \to F_m(R)$ can be uniquely extended to a derivation of $F_m(R)$.

Hint. Define ∂ by

$$\partial(x_{i_1}\dots x_{i_k})=\sum_{p=1}^{k}x_{i_1}\dots x_{i_{p-1}}\partial(x_{i_p})x_{i_{p+1}}\dots x_{i_k}.$$

Show that ∂ is well defined (i.e. $\partial(f)=0$ if $f=0$). Use that for any multihomogeneous $f\in F_m(R)$, we can express $\partial(f)$ as a sum of polynomials obtained by substitutions of $\partial(x_i)$ in the partial linearizations of f. Hence $f=0$ implies $\partial(f)=0$.

Definition 10.2.2 The derivation ∂ of the algebra $K[V_m]$, $K\langle V_m\rangle$ or $F_m(R)$ is called *locally nilpotent* if for any u in the algebra, there exists a positive integer n such that $\partial^n(u)=0$. The derivation is *triangular*, if $\partial(x_i)$ depends on $x_{i+1},\dots,x_m$ only, $i=1,\dots,m$.

Exercise 10.2.3 (i) Show that the derivation ∂ of $K[V_m]$, $K\langle V_m\rangle$ or $F_m(R)$ is locally nilpotent if and only if it is nilpotent on the generators of the algebra.

(ii) Show that every triangular derivation is locally nilpotent. In particular, the usual partial derivative $\partial/\partial x_j$ (defined by $\partial x_i/\partial x_j=1$ if $i=j$ and $\partial x_i/\partial x_j=0$ if $i\neq j$) is locally nilpotent.

Hint. (i) Use the *Leibniz formula*

$$\partial^n(u_1\dots u_k)=\sum\frac{n!}{n_1!\dots n_k!}\partial^{n_1}(u_1)\dots\partial^{n_k}(u_k),$$

where the summation runs on all $n_1,\dots,n_k$ such that $n_1+\dots+n_k=n$. Hence $\partial^n(x_{i_1}\dots x_{i_k})$ is a linear combination of $\partial^{n_1}(x_{i_1})\dots\partial^{n_k}(x_{i_k})$ with $n_1+\dots+n_k=n$. (ii) Use induction on m and show that for n large enough $\partial^n(x_{i_1}\dots x_{i_k})$ does not depend on x_1.

Exercise 10.2.4 (i) Let ∂ be a locally nilpotent derivation of $K[V_m]$, and let w be in the kernel of ∂. Show that the mapping $\Delta:u\to w\partial(u)$ is also a locally nilpotent derivation of $K[V_m]$.

(ii) Show that the statement of (i) is not always true when we replace $K[V_m]$ by $K\langle V_m\rangle$ or $F_m(R)$.

Hint. (i) Show that $\Delta(uv)=\Delta(u)v+u\Delta(v)$, $u,v\in K[V_m]$. (ii) Consider the derivation ∂ of $K\langle x,y,z\rangle$ defined by $\partial(x)=y$, $\partial(y)=\partial(z)=0$ and $w=z$. Show that $\Delta=z\partial$ satisfies

$$\Delta(x^2)=z(xy+yx),\ \Delta(x)x+x\Delta(x)=zyx+xzy.$$

Exercise 10.2.5 Show that the following mappings induce locally nilpotent derivations:

(i) $\partial(x)=y^2$, $\partial(y)=1$, for $K[x,y]$, $K\langle x,y\rangle$ and $F_2(R)$;

(ii) $\partial(x) = -2yw$, $\partial(y) = zw$, $\partial(z) = 0$, where $w = y^2 + xz$, for $K[x,y,z]$;
(iii) $\partial(x) = yw$, $\partial(y) = zw$, $\partial(z) = 0$, where $w = y^2 - 2xz$, for $K[x,y,z]$;
(iv) $\partial(x) = yw$, $\partial(y) = 0$, $\partial(z) = wy$, where $w = xy - yz$, for $K[x,y,z]$, $K\langle x,y,z\rangle$ or $F_3(R)$;
(v) $\partial(x) = wz$, $\partial(y) = tw$, $\partial(z) = \partial(t) = 0$, where $w = tx - yz$, for $K[x,y,z,t]$, $K\langle x,y,z,t\rangle$ or $F_4(R)$.

Hint. Show that in the case of polynomial algebras $\partial(w) = 0$ in (ii)–(v). For (relatively) free algebras use Exercise 10.2.3 (i).

Exercise 10.2.6 Show that the derivations in Exercise 10.2.5 (ii) and (iii) are of the form $\partial = w\partial_1$, where ∂_1 is a locally nilpotent derivation and $w \in \mathrm{Ker}\partial_1$.

Hint. In the notation of Exercise 10.2.5 (ii) and (iii), define ∂_1 respectively as

$$\partial_1(x) = -2y,\ \partial_1(y) = z,\ \partial_1(z) = 0,$$

$$\partial_1(x) = y,\ \partial_1(y) = z,\ \partial_1(z) = 0.$$

Definition 10.2.7 Let ϕ be an endomorphism of $K[V_m]$ and let $\partial/\partial x_p$ be the usual partial derivative with respect to x_p. The *Jacobian matrix* of ϕ is the $m \times m$ matrix with entries from $K[V_m]$

$$J(\phi) = \begin{pmatrix} \frac{\partial\phi(x_1)}{\partial x_1} & \frac{\partial\phi(x_2)}{\partial x_1} & \cdots & \frac{\partial\phi(x_m)}{\partial x_1} \\ \frac{\partial\phi(x_1)}{\partial x_2} & \frac{\partial\phi(x_2)}{\partial x_2} & \cdots & \frac{\partial\phi(x_m)}{\partial x_2} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial\phi(x_1)}{\partial x_m} & \frac{\partial\phi(x_2)}{\partial x_m} & \cdots & \frac{\partial\phi(x_m)}{\partial x_m} \end{pmatrix}.$$

Warning 10.2.8 In our notation, the (i,j)-entry of the Jacobian matrix of ϕ is $\partial\phi(x_j)/\partial x_i$. In commutative algebra one often denotes the endomorphisms by $f = (f_1, \ldots, f_m)$, using even capitals: $F = (F_1, \ldots, F_m)$, and the composition of f with $g = (g_1, \ldots, g_m)$ is defined by

$$f(g) = (f_1(g_1, \ldots, g_m), \ldots, f_m(g_1, \ldots, g_m)).$$

Then it is more convenient to define the (i,j)-entry of the Jacobian matrix $J(f)$ of f as $\partial f_i/\partial x_j$. Clearly, replacing our Jacobian matrix with its transpose would force changes also in other formulas, e.g. in the chain rule in the next exercise.

Exercise 10.2.9 (i) Show that the Jacobian matrix satisfies the *chain rule*: If ϕ and ψ are endomorphisms of $K[V_m]$, then

$$J(\phi \circ \psi) = J(\phi)\phi(J(\psi)),$$

where $\phi(J(\psi))$ means that we apply ϕ to the entries of $J(\psi)$.

(ii) Show that if ϕ is an automorphism of $K[V_m]$, then $J(\phi)$ is invertible.

Hint. (i) Let $\phi(x_i) = f_i$, $\psi(x_i) = g_i$, $i = 1, \ldots, m$. Then $(\phi \circ \psi)(x_i) = g_i(f_1, \ldots, f_m)$ and

$$\frac{\partial((\phi \circ \psi)(x_q))}{\partial x_p} = \frac{\partial g_q(f_1, \ldots, f_m)}{\partial x_p} =$$

$$= \sum_{k=1}^{m} \frac{\partial g_q(f_1, \ldots, f_m)}{\partial f_k} \frac{\partial f_k}{\partial x_p} = \sum_{k=1}^{m} \phi\left(\frac{\partial \psi(x_q)}{\partial x_k}\right) \frac{\partial \phi(x_k)}{\partial x_p}$$

and this implies the chain rule.

(ii) Use that the Jacobian matrix of $\phi \circ \phi^{-1}$ is the identity matrix, i.e. $J(\phi)$ is invertible.

Now we have the necessary background on derivations and start the proof of Makar-Limanov [179] of the theorem of Jung-Van der Kulk.

Definition 10.2.10 Let ∂ be a locally nilpotent derivation of an algebra R. We define a *degree function* $\deg_\partial$ on R by

$$\deg_\partial(u) = \min(n \mid \partial^n(u) = 0) - 1, \text{ if } 0 \neq u \in R.$$

We also assume that $\deg_\partial(0) = -\infty$.

For example, if $R = K[x]$ and ∂ is the usual derivative, we obtain the ordinary degree function: to make zero from a polynomial of degree 3 we should take four derivatives. For any derivation ∂, if $\partial(u) \neq 0$ then $\deg_\partial(\partial(u)) = \deg_\partial(u) - 1$.

Lemma 10.2.11 *If the algebra R has no zero divisors and if ∂ is a locally nilpotent derivation of R, then for every $u, v \in R$*

(i) $\deg_\partial(u + v) \leq \max(\deg_\partial(u), \deg_\partial(v))$,

(ii) $\deg_\partial(uv) = \deg_\partial(u) + \deg_\partial(v)$.

(iii) *If $uv \neq 0$ and $\partial(uv) = 0$, then $\partial(u) = \partial(v) = 0$.*

Proof. The inequality (i) is obvious because $\partial(u + v) = \partial(u) + \partial(v)$. The equality (ii) follows from the Leibniz formula

$$\partial^n(uv) = \sum_{k=0}^{n} \binom{n}{k} \partial^{n-k}(u)\partial^k(v).$$

If $p = \deg_\partial(u)$ and $q = \deg_\partial(v)$ then $\partial^{p+q+1}(uv) = 0$. On the other hand

$$\partial^{p+q}(uv) = \binom{p+q}{p} \partial^p(u)\partial^q(v) \neq 0.$$

The statement of (iii) follows from (ii) because

$$0 = \deg_\partial(uv) = \deg_\partial(u) + \deg_\partial(v),$$

where $\deg_\partial(u)$ and $\deg_\partial(v)$ are nonnegative integers. Therefore

$$\deg_\partial(u) = \deg_\partial(v) = 0.$$

Now we consider the polynomial algebra $K[x, y]$. Let

$$\text{Jac}(u, v) = u_x v_y - u_y v_x = \frac{\partial u}{\partial x}\frac{\partial v}{\partial y} - \frac{\partial u}{\partial y}\frac{\partial v}{\partial x}$$

be the *Jacobian* of (u, v), it is the determinant of the Jacobian matrix. With any polynomial f of $K[x, y]$ we associate a derivation as in the following lemma.

Lemma 10.2.12 *Let f be a fixed element of $K[x, y]$ and let $\partial(v) : K[x, y] \to K[x, y]$ be the mapping defined by*

$$\partial(v) = \text{Jac}(f, v), \ v \in K[x, y].$$

Then ∂ is a derivation of $K[x, y]$. If $f = \phi(x)$ and ϕ is an automorphism of $K[x, y]$, then ∂ is locally nilpotent.

Proof. It is clear from the definition of the Jacobian that for any fixed element $f \in K[x, y]$

$$\partial = f_x \frac{\partial}{\partial y} - f_y \frac{\partial}{\partial x}$$

is a derivation (see the hint to Exercise 2.1.15). Now, let $f = \phi(x)$, $g = \phi(y)$ for some automorphism ϕ of $K[x, y]$. By Exercise 10.2.9 (ii), the Jacobian matrix of ϕ is invertible, $0 \neq \alpha = \det(J(\phi)) \in K$ and $K[x, y] = K[f, g]$. For $w = f$ and $w = g$ we obtain that

$$\partial(f) = \text{Jac}(f, f) = 0, \ \partial(g) = \text{Jac}(f, g) = \alpha \in K, \ \partial^2(g) = 0.$$

Since the derivation ∂ acts nilpotently on the generators f and g of the algebra $K[f, g] = K[x, y]$, by Exercise 10.2.3 (i), it is locally nilpotent.

It is too difficult to describe the polynomials $\phi(x)$, where ϕ is an automorphism of $K[x, y]$. Instead we shall describe their top homogeneous components. We fix two positive relatively prime integers p and q and assign to x weight p and to y weight q. In this way, we give $K[x, y]$ the structure of a graded vector space. For every polynomial $v(x,y)$ we denote by $\overline{v}$ its *leading homogeneous component* with respect to this grading. If $v=\overline{v}$, then v is (p,q)*-homogeneous.*

For example, let $p=2$, $q=5$ and let

$$v(x,y)=5x^6y^3-xy^5+2x^{11}y+6x^4y^2-3x^{10}y.$$

Then $5x^6y^3$, $-xy^5$ and $2x^{11}y$ are of degree

$$6\times 2+3\times 5=1\times 2+5\times 5=11\times 2+1\times 5=27,$$

$6x^4y^2$ and $-3x^{10}y$ are respectively of degree $4\times 2+2\times 5=18$ and $10\times 2+1\times 5=25$. Hence

$$\overline{v(x,y)}=5x^6y^3-xy^5+2x^{11}y.$$

Lemma 10.2.13 (i) *Let* $u,v\in K[x,y]$*. If* $\mathrm{Jac}(\overline{u},\overline{v})\neq 0$*, then* $\overline{\mathrm{Jac}(u,v)}=\mathrm{Jac}(\overline{u},\overline{v})$*.*

(ii) *If f is a fixed element of $K[x,y]$ and the derivation of $K[x,y]$ defined by* $\partial(v)=\mathrm{Jac}(f,v)$*,* $v\in K[x,y]$*, is locally nilpotent, then the derivation* ∂_1 *defined by*

$$\partial_1(v)=\mathrm{Jac}(\overline{f},v),\ v\in K[x,y],$$

is also locally nilpotent.

Proof. (i) Since

$$\mathrm{Jac}(x^ay^b,x^cy^d)=(ad-bc)x^{a+c-1}y^{b+d-1},$$

and $\mathrm{Jac}(u,v)$ is a linear combination of $\mathrm{Jac}(x^ay^b,x^cy^d)$, we obtain that the leading homogeneous component of $\mathrm{Jac}(u,v)$ is not heavier than $\mathrm{Jac}(\overline{u},\overline{v})$ (with respect to our (p,q)-grading). If $\mathrm{Jac}(\overline{u},\overline{v})\neq 0$, then we have the required equality.

(ii) Let $\partial(v)=\mathrm{Jac}(f,v)$ be a locally nilpotent derivation. If u,v are (p,q)-homogeneous and $\mathrm{Jac}(u,v)\neq 0$, then $\mathrm{Jac}(u,v)$ is also (p,q)-homogeneous and this implies that $\partial_1^n(x)$ is (p,q)-homogeneous for all n and $\overline{\partial^n(x)}=\partial_1^n(x)$ if $\partial_1^n(x)\neq 0$. Since $\partial^n(x)=0$ for some n, we obtain that $\partial_1^k(x)=0$ for some $k\leq n$. Similar arguments for y give that $\partial_1^l(y)=0$ for some l. Hence ∂_1 is locally nilpotent by Exercise 10.2.3 (i).

Lemma 10.2.14 *If* $f(x,y)\in K[x,y]$ *is* (p,q)*-homogeneous then f can be decomposed in the form*

$$f(x,y)=\alpha x^ay^b\prod_{c=1}^{k}(x^q-\beta_cy^p),\ 0\neq\alpha\in K,\beta_c\in K,\ a,b,k\geq 0.$$

Proof. We write $f(x,y)$ in the form

$$f(x,y)=\alpha x^a y^b(x^{a_0}+\alpha_1 x^{a_1}y^{b_1}+\ldots+\alpha_{k-1}x^{a_{k-1}}y^{b_{k-1}}+\alpha_k y^{b_k}),$$

$\alpha,\alpha_i\in K$, $\alpha,\alpha_k\neq 0$. Since f is (p,q)-homogeneous, there exists a $d\geq 0$ such that

$$d=pa_0=pa_1+qb_1=\ldots=pa_{k-1}+qb_{k-1}=qb_k.$$

Since p and q are relatively prime, all $a_0,a_1,\ldots,a_{k-1}$ are divisible by q. Similarly all $b_1,\ldots,b_k$ are divisible by p. Hence

$$f(x,y)=x^a y^b(s^{c_0}+\alpha_1 s^{c_1}t^{d_1}+\ldots+\alpha_{k-1}s^{c_{k-1}}t^{d_{k-1}}+\alpha_k t^{d_k}),$$

where $s=x^q$, $t=y^p$ and

$$c_0=c_1+d_1=\ldots=c_{k-1}+d_{k-1}=d_k=\frac{a_0}{q}.$$

Since the field is algebraically closed, we can decompose $f(x,y)$ as desired.

In most of the proofs of the theorem of Jung-Van der Kulk one needs to show that the leading components of $\phi(x)$ and $\phi(y)$ with respect to some grading of $K[x,y]$ have a nice form for any automorphism ϕ of $K[x,y]$.

Lemma 10.2.15 *Let $\phi\in\mathrm{Aut}K[x,y]$ and let p,q be relatively prime positive integers. Then the leading component $\overline{\phi(x)}$ of $\phi(x)$ with respect to the (p,q)-grading of $K[x,y]$ has the form αx^i, αy^j or $\alpha(x^q-\beta y^p)^k$, for some $\alpha,\beta\in K$, $\alpha\neq 0$.*

Proof. By Lemma 10.2.14,

$$\overline{\phi(x)}=\alpha x^a y^b\prod_{c=1}^{k}(x^q-\beta_c y^p),\ 0\neq\alpha\in K,\ \beta_c\in K.$$

We consider the derivations ∂ and ∂_1 of $K[x,y]$ defined by

$$\partial(v)=\mathrm{Jac}(\phi(x),v),\ \partial_1(v)=\mathrm{Jac}(\overline{\phi(x)},v),\ v\in K[x,y].$$

Since $\partial_1(\overline{\phi(x)})=0$, according to Lemma 10.2.11 (iii), all factors of $\overline{\phi(x)}$ are ∂_1-constants. If we have two essentially different (algebraically independent) factors, say $x^q-\beta_1 y^p$ and $x^q-\beta_2 y^p$, then $\partial_1(x^q)=\partial_1(y^p)=0$. Applying once again Lemma 10.2.11 (iii), we obtain that $\partial_1(x)=\partial_1(y)=0$ and, hence, $\partial_1=0$. This is impossible, because ∂_1 is not the zero derivation.

Lemma 10.2.16 *Let $\phi\in\mathrm{Aut}K[x,y]$. If the leading component $\overline{\phi(x)}$ of $\phi(x)$ with respect to the (p,q)-grading of $K[x,y]$ has the form $\alpha(x^q-\beta y^p)^k$, and $\beta\neq 0$, then $p=1$ or $q=1$.*

Proof. If $\overline{\phi(x)}=(x^q-\beta y^q)^k$ then, by Lemmas 10.2.12 and 10.2.13 (ii), the derivation ∂_1 of $K[x,y]$ defined by $\partial_1(v)=\mathrm{Jac}((x^q-\beta y^p)^k,v)$, $v\in K[x,y]$, is locally nilpotent. Let ∂_2 be the derivation defined by $\partial_2(v)=\mathrm{Jac}(x^q-\beta y^p,v)$, $v\in K[x,y]$. Then $x^q-\beta y^p\in\mathrm{Ker}\partial_2$ and $\partial_1=k(x^q-\beta y^p)^{k-1}\partial_2$. Hence, as in Exercise 10.2.4 (i),

$$\partial_1^n=(k(x^q-\beta y^p)^{k-1})^n\partial_2^n$$

and ∂_2 is also locally nilpotent. Now $\partial_2(x)=\beta py^{p-1}$ and $\partial_2(y)=qx^{q-1}$. If we consider the degree function $\deg=\deg_{\partial_2}$ relative to ∂_2 (see Definition 10.2.10) and denote by $d_x=\deg(x)$ and $d_y=\deg(y)$ then

$$d_x-1=(p-1)d_y,\ d_y-1=(q-1)d_x.$$

Hence $-2=(q-2)d_x+(p-2)d_y$. Since p and q are positive integers, it is possible only if p or q is equal to 1.

Now we prove the theorem which describes the automorphisms of $K[x,y]$.

Theorem 10.2.17 (Jung [130] and Van der Kulk [257]) *Every automorphism of* $K[x,y]$ *is tame.*

Proof. Let $\phi\in\mathrm{Aut}K[x,y]$. As usually, we denote by u_x and u_y the partial derivatives of u with respect to x and y. We apply induction on the product $\deg_x\phi(x)\cdot\deg_y\phi(x)$. The base of the induction is when this product is 0, i.e. $\phi(x)$ does not depend on one of the variables x and y. If $\phi(x)$ depends only on x, then

$$\mathrm{Jac}(\phi(x),\phi(y))=\phi(x)_x\phi(y)_y$$

is a nonzero constant. Therefore $\phi(x)_x$ and $\phi(y)_y$ are also nonzero constants and

$$\phi(x)=\alpha x+\beta,\ \phi(y)=\gamma y+p(x),\ \alpha,\beta,\gamma\in K,\alpha,\gamma\neq 0,\ p(x)\in K[x].$$

Hence ϕ is a triangular automorphism. If $\phi(x)$ depends only on y, then

$$\phi(x)=\alpha y+\beta,\ \phi(y)=\gamma x+p(y),\ \alpha,\beta,\gamma\in K,\alpha,\gamma\neq 0,\ p(y)\in K[y].$$

In this case $\phi=\psi\circ\tau$, where ψ is a triangular automorphism and τ is the automorphism defined by $\tau(x)=y$, $\tau(y)=x$, i.e. τ changes the places of x and y. Hence, in both the cases, ϕ is tame. Now, let $\phi(x)$ depend on both x and y. Let

$$\deg_x\phi(x)=a>0,\ \deg_y\phi(x)=b>0.$$

We choose relatively prime positive integers p and q such that $pa=qb$ and introduce the (p,q)-grading on $K[x,y]$. Clearly, the leading (p,q)-homogeneous component $\overline{\phi(x)}$ of $\phi(x)$ depends on both x and y. (Why? Answer: If the (p,q)-degree of $\phi(x)$ is equal to ap, then $\overline{\phi(x)}$ contains as a summand the monomial of $\phi(x)$ of maximal degree a in x as well as the monomial of maximal degree b in y. If the (p,q)-degree of $\phi(x)$ is bigger than ap, then each monomial of $\overline{\phi(x)}$ depends both on x and y.) By Lemma 10.2.16, $p=1$ or $q=1$ and $\overline{\phi(x)}$ is either of the form $\alpha(y-\beta x^q)^k$ (when $p=1$) or of the form $\alpha(x-\beta y^p)^k$ (when $q=1$), in both the cases $0\neq\alpha,\beta\in K$. If $\overline{\phi(x)}=\alpha(y-\beta x^q)^k$, let us compose ϕ with the tame automorphism ψ defined by $\psi(x)=x$, $\psi(y)=y+\beta x^q$. Then $\overline{(\psi\circ\phi)(x)}=\alpha y^k$. Now

$$\deg_x(\psi\circ\phi)(x)<\deg_x\phi(x),\ \deg_y(\psi\circ\phi)(x)=\deg_y\phi(x)$$

(why?). Similarly, if $\overline{\phi(x)}=\alpha(x-\beta y^p)^k$, then we compose ϕ with the tame automorphism ψ given by $\psi(x)=x+\beta y^p$, $\psi(y)=y$. Now $\overline{(\psi\circ\phi)(x)}=\alpha x^k$ and

$$\deg_x(\psi\circ\phi)(x)=\deg_x\phi(x),\ \deg_y(\psi\circ\phi)(x)<\deg_y\phi(x).$$

By inductive arguments, the automorphism $\psi\circ\phi$ is tame and, since ψ is also tame, we obtain the tameness of ϕ.

The group $\mathrm{Aut}K[x,y]$ has a description in terms of combinatorial group theory which follows from the careful study of the proof of Van der Kulk [257]. A version of the proof can be found in the book by Cohn [51] or in the paper by Dicks [56]. See also the important paper by Wright [267] which contains also results for the automorphism group of the C-algebra $\mathrm{Aut}C[x,y]$, where C is a commutative domain.

Definition 10.2.18 Let G_1 and G_2 be two groups, containing respectively subgroups H_1 and H_2 isomorphic to a given group H. The *free product of G_1 and G_2 amalgamating H* is the group $G_1*_H G_2$ with generating set consisting on the disjoint union of generating sets of G_1 and G_2, with defining relations consisting of the relations of G_1, G_2 and of relations identifying the corresponding elements of H_1 and H_2. In other words, if G_1 and G_2 are generated respectively by $\{g'_p\mid p\in P\}$ and $\{g''_q\mid q\in Q\}$ with defining relations respectively R_1 and R_2, and $\nu_i:H\to H_i$ is the isomorphism of H and H_i, $i=1,2$, then $G_1*_H G_2$ is generated by $\{g'_p,g''_q\mid p\in P,q\in Q\}$ and has defining relations $R_1\cup R_2\cup\{\nu_1(h)=\nu_2(h)\mid h\in H\}$.

Example 10.2.19 Let $G_1=\langle x\mid x^4=1\rangle$ be the cyclic group of order 4, let $G_2=\langle y,z\mid y^2=z^3=yzyz=1\rangle$ be the symmetric group S_3, where we identify y with a transposition and z with a cycle of length 3. Let H be the cyclic group of order 2 embedded into G_1 and G_2 by $H_1=\langle x^2\rangle$, $H_2=\langle y\rangle$. Then $G_1*_H G_2$ is the group generated by x,y,z (the generators of G_1 and G_2) and with defining relations

$$x^4=1\ \text{(relations of } G_1\text{)},\ y^2=z^3=yzyz=1\ \text{(relations of } G_2\text{)},$$

$$y=x^2\ \text{(identification of the elements of } H_1 \text{ and } H_2\text{)}.$$

For example, we calculate the element $xyzyzyx^3zx$ of $G_1*_H G_2$. First, working in G_2 we present $yzyzy$ as $(yzyz)y=y$ and

$$xyzyzyx^3zx=x(yzyzy)x^3zx=xyx^3zx.$$

Using that $y = x^2$, we see that

$$xyx^3zx = xx^2x^3zx = x^6zx = x^2zx.$$

For the properties of amalgamated free products of groups see e.g. the book by Magnus, Karrass and Solitar [176] (or, for the minimum needed to read the proof of the next theorem, the paper by Wright [267]). The combinatorial description of $\mathrm{Aut}K[x,y]$ mentioned above is the following.

Theorem 10.2.20 *The group* $\mathrm{Aut}K[x,y]$ *is the amalgamated free product of the group of affine automorphisms and the group of triangular automorphisms over their intersection with the triangular affine automorphisms. In other words, if* G_1 *is the affine group of the two-dimensional vector space with basis* $\{x,y\}$ *and* G_2 *is the group of triangular automorphisms* $(\alpha_1x+f(y),\alpha_2y+\beta_2)$, $0\neq\alpha_1,\alpha_2\in K$, $\beta_2\in K$, $f(y)\in K[y]$, *and* H *is the intersection of these two groups,*

$$H = \{(\alpha_1x+\beta_1y+\gamma,\alpha_2y+\beta_2)\mid 0\neq\alpha_1,\alpha_2\in K,\beta_1,\beta_2,\gamma\in K\},$$

then $\mathrm{Aut}K[x,y] = G_1 *_H G_2$.

An equivalent form of this theorem in the language of IL-automorphisms is given by Shpilrain and Yu [242]. We give a weaker version of their result.

Theorem 10.2.21 (Shpilrain and Yu [242]) *Let* G *be the group of IL-automorphisms of* $K[x,y]$, *i.e. the automorphisms* ϕ *of* $K[x,y]$ *such that*

$$\phi(x) = x+\ldots,\ \phi(y) = y+\ldots,$$

where $\ldots$ *means a sum of monomials of higher degree. Considering* G *as a subgroup of the group* $\mathrm{Aut}^0K[x,y]$ *of augmentation preserving automorphisms,* G *is the normal closure in* $\mathrm{Aut}^0K[x,y]$ *of the subgroup* H *generated by the elementary automorphisms*

$$\phi_f = (x+f(y),y),\ \psi_g = (x,y+g(x)),$$

where $f(y)$ *and* $g(x)$ *are polynomials without constant and linear terms. The group* H *is isomorphic to the free product of the abelian groups*

$$H_x = \{\phi_f = (x+f(y),y)\in G\},\ H_y = \{\psi_g = (x,y+g(x))\in G\}.$$

In other words, every automorphism η *in* H *can be uniquely presented as a product*

$$\eta = \phi_{f_1}^{\varepsilon_1}\circ\psi_{g_1}\circ\phi_{f_2}\circ\psi_{g_2}\circ\ldots\circ\phi_{f_k}\circ\psi_{g_k}^{\varepsilon_2},\ \varepsilon_i = 0,1,\ i = 1,2,$$

and every automorphism ϕ *in* G *has a presentation*

$$\phi=(\sigma_1\eta_1\sigma_1^{-1})(\sigma_2\eta_2\sigma_2^{-1})\dots(\sigma_p\eta_p\sigma_p^{-1}),\ \eta_j\in H,\ \sigma_j\in\mathrm{Aut}^0K[x,y].$$

Exercise 10.2.22 Show that the groups H_x and H_y in Theorem 10.2.21 are "naturally" isomorphic to the additive groups of all polynomials in one variable without constant and linear terms.

Hint. Show that the composition of $(x+f_1(y),y)$ and $(x+f_2(y),y)$ is equal to $(x+f_1(y)+f_2(y),y)$.

10.3 The Free Associative Algebra of Rank Two

In this section we shall prove the theorem of Makar-Limanov [177] and Czerniakiewicz [53] describing the automorphisms of $K\langle x,y\rangle$. Proofs can be also found in the book of Cohn [51] or in the paper of Dicks [57]. Our proof is based on the proof in [51].

Definition 10.3.1 Let p,q be two integers, not both equal to zero.

(i) We define a (p,q)-bidegree $D:K\langle x,y\rangle\to\mathbb{Z}^2\cup\{-\infty\}$ assuming that the bidegree of x is equal to $(1,0)$, the bidegree of y is $(0,1)$ and $D(0)=-\infty$. (Hence D is the usual bidegree on $K\langle x,y\rangle$. A monomial u is of (p,q)-bidegree equal to (a_1,a_2) if x appears a_1 times in u and y appears a_2 times.)

(ii) We introduce a linear ordering on $\mathbb{Z}^2$: If $(a_1,a_2),(b_1,b_2)\in\mathbb{Z}^2$, then $(a_1,a_2)\prec(b_1,b_2)$ if and only if $a_1p+a_2q<b_1p+b_2q$ or, if $a_1p+a_2q=b_1p+b_2q$ then $a_1q<b_1q$.

(iii) For the polynomial $f(x,y)\in K\langle x,y\rangle$ we denote by $\mathrm{supp}(f)$ the set of all points (a_1,a_2) in $(\mathbb{N}\cup\{0\})\times(\mathbb{N}\cup\{0\})$ such that f contains monomials of (p,q)-bidegree (a_1,a_2). The *$\prec$-leading component* $\overline{f}$ of f is the sum of all monomials of f which are of maximal degree with respect to $\prec$.

(iv) We also define a (p,q)-degree function $d:K\langle x,y\rangle\to\mathbb{Z}\cup\{-\infty\}$ assuming that $d(x)=p$, $d(y)=q$.

Example 10.3.2 Let $p=3$, $q=-2$. Then

$$(2,4)\prec(5,6)\prec(3,3),$$

because

$$2\cdot3+4\cdot(-2)=-2<5\cdot3+6\cdot(-2)=3\cdot3+3\cdot(-2)=3,\ 5\cdot(-2)<3\cdot(-2).$$

Hence, for

$$f=2yx^2yxy-x^2y^3x-x^4y^6x+yx^2y^3+2xyx^2y^2$$

the $\prec$-leading component is

$$\overline{f}=2yx^2yxy-x^2y^3x+2xyx^2y^2.$$

Exercise 10.3.3 Show that the homomorphism of additive groups

$$\lambda:\mathbb{Z}^2\longrightarrow\mathbb{Z},\lambda:(a_1,a_2)\to a_1p+a_2q,$$

preserves the grading of Definition 10.3.1 (ii), i.e. if $a=(a_1,a_2)$, $b=(b_1,b_2)$ and $a\prec b$, then $\lambda(a)\leq\lambda(b)$.

Exercise 10.3.4 Let p and q be integers, not both equal to zero, and let f and g be two polynomials in $K\langle x,y\rangle$ such that their leading components $\overline{f}$ and $\overline{g}$ with respect to the (p,q)-bigrading are *independent*, i.e. generate a free subalgebra of $K\langle x,y\rangle$. Show that f and g generate a free subalgebra of $K\langle x,y\rangle$, the leading terms of the polynomials in $K\langle f,g\rangle$ are contained in the free subalgebra generated by $\overline{f}$ and $\overline{g}$ and

$$\{D(h)\mid h\in K\langle f,g\rangle\}\subseteq(\mathbb{N}\cup\{0\})\cdot D(f)+(\mathbb{N}\cup\{0\})\cdot D(g)\cup\{-\infty\}.$$

Hint. Use that the (p,q)-bigrading of $K\langle x,y\rangle$ satisfies the conditions

$$D(u)D(v)=D(uv),\ D(u+v)\preceq\max(D(u),D(v)),\ u,v\in K\langle x,y\rangle.$$

Derive from here the statement of the exercise.

Lemma 10.3.5 *Let $f,g\in K\langle x,y\rangle$ be homogeneous with respect to the (p,q)-bigrading of $K\langle x,y\rangle$. Then either f and g generate a free subalgebra of $K\langle x,y\rangle$ or, up to multiplicative constants, they both are powers of the same homogeneous element of $K\langle x,y\rangle$.*

Proof. Without loss of generality we may assume that f and g are not constants in the field K. Since the (p,q)-homogeneous elements of $K\langle x,y\rangle$ are multihomogeneous in the usual sense, we obtain that $f(0,0)=g(0,0)=0$. Let f and g be dependent, i.e. they do not generate a free algebra. Therefore $h(f,g)=0$ for some polynomial $0\neq h(t,u)\in K\langle t,u\rangle$. Then

$$0=h(f(0,0),g(0,0))=h(0,0).$$

Hence $h(t,u)$ has no constant term, $h(t,u)$ has the form

$$h(t,u)=th_1(t,u)+uh_2(t,u),$$

where $h_1(t,u),h_2(t,u)\in K\langle t,u\rangle$, and

$$h(f,g)=fh_1(f,g)+gh_2(f,g)=0.$$

Let the length of the monomials of f be bigger than or equal to the length of the monomials of g. Then $f=gf_1$ for some $f_1\in K\langle x,y\rangle$. (This fact is not trivial. If some problems arise, see Section 6.7 of the book by Cohn [51].) Hence

$$0 = h(f, g) = gf_1h_1(gf_1, g) + gh_2(gf_1, g) = gh'(f_1, g).$$

Since the free algebra $K\langle x, y\rangle$ has no zero divisors, we obtain that $h'(f, g) = 0$ and, by induction, $f_1 = \alpha v^k$, $g = \beta v^l$, where $\alpha, \beta \in K$ and v is a homogeneous polynomial in $K\langle x, y\rangle$.

Proposition 10.3.6 *Let f and g be two polynomials of $K\langle x, y\rangle$ such that f is congruent to x and g is congruent to y modulo the commutator ideal of $K\langle x, y\rangle$. Let $f \neq x$ or $g \neq y$. Then there exist two integers p and q, not both equal to zero, such that $p \leq 0 \leq q$,*

$$D(f) + D(g) \neq D(x) + D(y) = (1, 1)$$

in the (p, q)-bigrading of $K\langle x, y\rangle$,

$$(d(f), d(g)) = (d(x), d(y)) = (p, q)$$

in the (p, q)-grading and $\overline{f}$ and $\overline{g}$ are independent. Moreover, $\phi = (f, g)$ is not an automorphism of $K\langle x, y\rangle$ and $[f, g] \neq \alpha[x, y]$, for all nonzero constants $\alpha \in K$.

Proof. Let $\text{supp}(f)$ be the set of all pairs (a_1, a_2) such that there exists a monomial of f of degree a_1 and a_2, respectively to x and to y. Since

$$f = x + \sum \alpha_i u_i[x, y]v_i,\ g = y + \sum \beta_j u'_j[x, y]v'_j,$$

for some $\alpha_i, \beta_j \in K$ and some monomials $u_i, v_i, u'_j, v'_j \in K\langle x, y\rangle$, all monomials of f contain x and all monomials of g contain y. Let us denote

$$\text{supp}(x^{-1}f) = \{(a_1 - 1, a_2) \mid (a_1, a_2) \in \text{supp}(f)\},$$

$$\text{supp}(y^{-1}g) = \{(b_1, b_2 - 1) \mid (b_1, b_2) \in \text{supp}(g)\}.$$

Clearly, the union S of $\text{supp}(f)$ and $\text{supp}(g)$, with the origin removed, lies in the first quadrant. If S contains a point $(0, p_2)$ with $p_2 \neq 0$, we choose $p = -p_2$, $q = 0$. If S contains no such a point, we choose $(p_1, p_2) \in S$, $(p_1, p_2) \neq (0, 0)$, such that the quotient p_2/p_1 is maximal and put $p = -p_2$, $q = p_1$.

Now we consider the ordering $\prec$ induced by p and q. Let us assume that $q > 0$, the case $q = 0$ is similar. By the choice of p and q, for a point (a_1, a_2) of S with $a_2/a_1 < p_2/p_1$ we obtain that $-a_1p_2 + a_2p_1 = a_1p + a_2q < 0$ and $(a_1, a_2) \prec (0, 0)$. If $a_2/a_1 = p_2/p_1$, then $a_2q > 0$ and $(0, 0) \prec (a_1, a_2)$. Hence all "positive" points of S lie on the half-line L from the origin $(0, 0)$ through $(p_1, p_2) = (q, -p)$. Now $\text{supp}(x^{-1}f)$ contains the origin, which is also on L. Hence the (p, q)-bidegrees of f and x satisfy $D(x) \preceq D(f)$ and $D(f) - D(x) \in L$. Similarly, $D(g) - D(y) \in L$. By the choice of p and

q, at least one of $D(f)-D(x)$ and $D(g)-D(y)$ is different from $(0,0)$. Hence $D(x)+D(y) \prec D(f)+D(g)$. Besides, for the (p,q)-degree we obtain $d(f)-d(x)=0$ (because $D(f)-D(x)\in L$), similarly $d(g)-d(y)=0$, i.e. $(d(f),d(g))=(d(x),d(y))=(p,q)$. If the leading components $\overline{f}$ and $\overline{g}$ are dependent, then by Lemma 10.3.5, up to multiplicative constants, they are positive powers of a homogeneous element v of $K\langle x,y\rangle$. Hence both $d(f)$ and $d(g)$ are positive multiples of $d(v)$ which is impossible, because $d(f)=d(x)=p<0\leq q=d(y)=d(g)$. Therefore $\overline{f}$ and $\overline{g}$ are independent.

Let $\phi=(f,g)$ be an automorphism of $K\langle x,y\rangle$. Then f and g generate $K\langle x,y\rangle$ and $x,y\in K\langle f,g\rangle$. Hence

$$D(x)=(1,0), D(y)=(0,1)\in\{D(u)\mid u\in K\langle f,g\rangle\}.$$

Since $\overline{f}$ and $\overline{g}$ are independent, by Exercise 10.3.4 we obtain that

$$\mathbb{N}^2\subseteq\{D(u)\mid u\in K\langle f,g\rangle\}\subseteq(\mathbb{N}\cup\{0\})\cdot D(f)+(\mathbb{N}\cup\{0\})\cdot D(g)\cup\{-\infty\}.$$

But this is impossible, because $(1,1)\prec D(f)+D(g)$ implies $\{D(f),D(g)\}\neq\{D(x),D(y)\}$ and $(\mathbb{N}\cup\{0\})\cdot D(f)+(\mathbb{N}\cup\{0\})\cdot D(g)$ does not cover the whole first quadrant. Hence $\phi=(f,g)$ is not an automorphism.

Finally, since $\overline{f}$ and $\overline{g}$ are independent,

$$\overline{fg}=\overline{f}\,\overline{g}\neq\overline{g}\,\overline{f}=\overline{gf}$$

and hence we obtain for the (p,q)-bigrading that

$$D([f,g])=\max(D(fg),D(gf))=D(f)+D(g)\succ D(x)+D(y)=D([x,y]).$$

This means that $[f,g]\neq\alpha[x,y]$ for $0\neq\alpha\in K$.

The following theorem is the main result of the section. It can be viewed as a noncommutative analogue of Jung-Van der Kulk Theorem 10.2.17.

Theorem 10.3.7 (Makar-Limanov [177] and Czerniakiewicz [53]) *All automorphisms of the free algebra $K\langle x,y\rangle$ are tame.*

Proof. There exists a natural homomorphism

$$\pi:\mathrm{Aut}K\langle x,y\rangle\longrightarrow\mathrm{Aut}K[x,y].$$

For every automorphism ϕ of $K\langle x,y\rangle$ the automorphism $\pi(\phi)$ of $K[x,y]$ sends x and y respectively to $\phi(x)$ and $\phi(y)$ modulo the commutator ideal of $K\langle x,y\rangle$. Every affine or triangular automorphism of $K[x,y]$ can be obviously lifted to an affine or triangular automorphism of $K\langle x,y\rangle$. By Theorem 10.2.17, every automorphism of $K[x,y]$ is tame, i.e. a product of affine and triangular automorphisms. Hence the homomorphism π is an epimorphism and every automorphism of $K[x,y]$ is induced by some automorphism of the free algebra $K\langle x, y\rangle$. We shall prove the theorem if we establish that the kernel of π is trivial. Let $\phi = (f, g) \in \mathrm{Ker}\pi$. Hence,

$$\phi(x) = f = x + \sum \alpha_i u_i[x, y]v_i,\ \phi(y) = g = y + \sum \beta_j u'_j[x, y]v'_j,$$

where $\alpha_i, \beta_j \in K,\ u_i, v_i, u'_j, v'_j \in K\langle x, y\rangle$. By Proposition 10.3.6, if $f \neq x$ or if $g \neq y$, then ϕ cannot be an automorphism. Hence, the kernel of ϕ is trivial and every automorphism of $K\langle x, y\rangle$ is a product of affine and triangular automorphisms.

Corollary 10.3.8 *The groups* $\mathrm{Aut}K\langle x, y\rangle$ *and* $\mathrm{Aut}K[x, y]$ *are isomorphic in a canonical way.*

Proof. In the notation of the proof of Theorem 10.3.7, the group homomorphism

$$\pi : \mathrm{Aut}K\langle x, y\rangle \longrightarrow \mathrm{Aut}K[x, y]$$

is an epimorphism with trivial kernel, i.e. π is an isomorphism.

It is easy to recognize whether an endomorphism of $K\langle x, y\rangle$ is an automorphism since there is a simple commutator test.

Theorem 10.3.9 (Dicks [55], Czerniakiewicz [53]) *An endomorphism* $\phi = (f, g)$ *of* $K\langle x, y\rangle$ *is an automorphism if and only if there exists a nonzero constant* $\alpha \in K$ *such that*

$$\phi([x, y]) = [f, g] = \alpha[x, y].$$

The next exercise and Proposition 10.3.6 give some idea about the proof of Theorem 10.3.9, although the complete proof needs additional essential work. The proof can be found in the paper by Dicks [55] or in the book by Cohn [51].

Exercise 10.3.10 Show that for every automorphism ϕ of $K\langle x, y\rangle$ there exists a nonzero constant $\alpha \in K$ such that

$$\phi([x, y]) = \alpha[x, y].$$

Hint. For an affine or for a triangular automorphism ϕ of $K\langle x, y\rangle$ it is easy to see that $\phi([x, y]) = \alpha[x, y]$, $0 \neq \alpha \in K$. Hence every automorphism, as a product of affine and triangular automorphisms, satisfies the same property for some $\alpha \in K$.

Concerning the automorphisms of the free associative algebra $K\langle V_m\rangle$, the situation is similar to that for the automorphisms of the polynomial algebra but the study in the noncommutative case seems to be less intensive than in the commutative. The problem whether all automorphisms are tame is still open for $m > 2$. One of the main difficulties is that the proof of Theorem 10.3.7 is based on the proof of Theorem 10.2.17 and we do not know the structure of $\mathrm{Aut} K[V_m]$ for $m > 2$.

10.4 Exponential Automorphisms

In this section we consider a class of automorphisms which appears naturally also in other algebraic considerations. We assume that the characteristic of K is equal to 0 and denote by R any PI-algebra.

Exercise 10.4.1 If ∂ is a locally nilpotent derivation, show that the mapping of $K[V_m]$, $K\langle V_m\rangle$ or $F_m(R)$

$$(\exp\partial)(u) = u + \frac{\partial(u)}{1!} + \frac{\partial^2(u)}{2!} + \dots$$

is well defined and is an automorphism of the algebra.

Hint. Use that the high powers ∂^n annihilate u and hence $\exp\partial$ is well defined. Use the Leibniz formula

$$\partial^n(uv) = \sum_{k=0}^{n} \binom{n}{k} \partial^k(u)\partial^{n-k}(v)$$

to show that $(\exp\partial)(uv) = (\exp\partial)(u)(\exp\partial)(v)$.

Definition 10.4.2 If ∂ is a locally nilpotent derivation of $K[V_m]$, $K\langle V_m\rangle$ or $F_m(R)$, we call $\exp\partial$ the *exponential automorphism induced by* ∂.

Exercise 10.4.3 (see e.g. Daigle and Freudenburg [54]) Show that u is in the kernel of the locally nilpotent derivation ∂ if and only if $(\exp\partial)(u) = u$.

Hint. Use Vandermonde arguments for

$$u = \exp^n\partial(u) = \exp(n\partial)(u) = u + n\frac{\partial(u)}{1!} + n^2\frac{\partial^2(u)}{2!} + \dots,\ n = 0, 1, 2, \dots$$

and derive that $\exp^n\partial(u) = u$ implies that $\partial(u) = 0$.

In 1970 Nagata [190] gave his famous example of an automorphism of $K[x, y, z]$ which fixes the variable z and is not tame considered as an automorphism of the $K[z]$-algebra $(K[z])[x, y]$. The automorphism of Nagata can be obtained in the following way.

Exercise 10.4.4 Let ∂ and Δ be derivations of $K[x,y,z]$ defined by

$$\partial(x) = -2y,\ \partial(y) = z,\ \partial(z) = 0,$$

$$\Delta(x) = -2yw,\ \Delta(y) = zw,\ \Delta(z) = 0,\ w = y^2 + xz.$$

Show that ∂ and $\Delta = w\partial$ are locally nilpotent and $\exp\Delta$ is the automorphism of $K[x,y,z]$ defined as

$$\nu = (x - 2y(y^2 + xz) - z(y^2 + xz)^2,\ y + z(y^2 + xz),\ z).$$

This is the *Nagata automorphism.*

Hint. Use Exercises 10.2.5, 10.2.6 and 10.4.3.

Exercise 10.4.5 Let

$$\phi_v = (x - 2vy - v^2 z,\ y + vz,\ z),\ v \in K[x,y,z],$$

be an endomorphism of $K[x,y,z]$. Show that ϕ_v is an automorphism if and only if $v = v_1(y^2 + xz, z)$, for some $v_1 = v_1(t_1,t_2) \in K[t_1,t_2]$. In this way one obtains a family of "Nagata like" automorphisms.

Hint. Use the previous Exercise 10.4.4. If in the notation there $v \in \mathrm{Ker}\partial$, then $\phi_v = \exp(v\partial)$ is an automorphism. If ϕ_v is an automorphism, its Jacobian matrix is invertible and its determinant is a nonzero scalar in K. Calculate the determinant of the Jacobian matrix of ϕ_v:

$$\det(J(\phi_v)) = 1 + 2yv_x + zv_y = 1,$$

where v_x and v_y are the partial derivatives of v. Show that the only solutions in $K[x,y,z]$ of the partial differential equation $2yv_x + zv_y = 0$ are the polynomials $v_1(y^2 + xz, z)$, where $v_1(t_1,t_2) \in K[t_1,t_2]$.

Conjecture 10.4.6 (Nagata [190]) The Nagata automorphism of $K[x,y,z]$ is wild.

There exist many evidences that the behaviour of the Nagata automorphism is different from this of the most tame automorphisms. See for example the book by Nagata [190] and the papers of Alev [8], Drensky, Gutierrez and Yu [90] and Le Bruyn [170].

Definition 10.4.7 An automorphism ϕ of $K[V_m]$ (respectively of $K\langle V_m\rangle$ or of $F_m(R)$ for some PI-algebra R) is called *stably tame* if there exists a positive integer p such that the extension of ϕ to $K[V_{m+p}]$ (respectively to $K\langle V_{m+p}\rangle$ or to $F_{m+p}(R)$) by $\phi(x_{m+k}) = x_{m+k}$, $k = 1,\dots,p$, is a tame automorphism.

Theorem 10.4.8 (Martha Smith [244]) *If ∂ is a triangular derivation of $K[V_m]$ and w is in the kernel of ∂, then the exponential automorphism $\exp(w\partial)$ is stably tame.*

Proof. We extend the derivation ∂ to $K[V_{m+1}]$ by $\partial(x_{m+1}) = 0$. Clearly, ∂ is still locally nilpotent, $x_{m+1} \in \mathrm{Ker}\partial$ and (prove it!) $\exp(x_{m+1}\partial)$ is a triangular automorphism of $K[V_{m+1}]$. Let τ be the triangular automorphism of $K[V_{m+1}]$ defined by

$$\tau(x_{m+1}) = x_{m+1} + w,\ \tau(x_i) = x_i,\ i = 1, \ldots, m.$$

Concrete calculations show that

$$\exp(w\partial) = \tau^{-1} \circ \exp(-x_{m+1}\partial) \circ \tau \circ \exp(x_{m+1}\partial)$$

and $\exp(w\partial)$ is a tame automorphism of $K[V_{m+1}]$. Since $\exp(w\partial)(x_{m+1}) = x_{m+1}$, the automorphism $\exp(w\partial)$ of $K[V_m]$ is stably tame.

Corollary 10.4.9 (Martha Smith [244]) *The Nagata automorphism is stably tame.*

Proof. By Exercise 10.4.4, the Nagata automorphism is equal to $\exp(w\partial)$, where

$$\partial(x) = -2y,\ \partial(y) = z,\ \partial(z) = 0,\ w = y^2 + xz.$$

Clearly, ∂ is a triangular derivation, $w \in \mathrm{Ker}\partial$ and we apply Theorem 10.4.8.

From some point of view, all known automorphisms of the polynomial algebra arise from natural constructions and are “nice looking”. Now we can make Problem 1.1.1 precise in the following way.

Problem 10.4.10 Is the automorphism group of $K[V_m]$, $m > 2$, generated by:

(i) automorphisms $\exp(\Delta)$, where $\Delta = w\partial$, $w \in \mathrm{Ker}\partial$ and ∂ is a triangular derivation;

(ii) automorphisms $\exp(\Delta)$, where Δ is a locally nilpotent derivation;

(iii) stably tame automorphisms;

(iv) other natural automorphisms?

The above discussion suggests the following question.

Problem 10.4.11 (Bergman, private communication) Can every nilpotent derivation of $K[V_m]$ be obtained by starting with a locally nilpotent derivation ∂ of some particular simple sort, multiplying it by a member w from the kernel of ∂ and conjugating by an automorphism of $K[V_m]$?

Remark 10.4.12 It is interesting to have more examples of naturally arising Nagata like automorphisms. This would provide new potential candidates for wild automorphisms of $K[V_m]$. Some wild automorphisms of the C-algebra $C[x_1,\dots,x_m]$, where C is any integral domain, are given in the book by Nagata [190] and in the paper by Wright [267]. Up till now, all stably tame automorphisms of $K[V_m]$ have been found using locally nilpotent derivations. For example, recently Van den Essen [103] and Drensky and Stefanov [97] have constructed a family of exponential automorphisms. The proof that the automorphisms in [103] and [97] are stably tame is similar to that of Theorem 10.4.8, but instead of triangular derivations uses the theorem of Suslin [248] that for $k \geq 3$ every invertible $k \times k$ matrix with entries from $K[V_m]$ is a product of elementary matrices. We recommend the book by Nowicki [196] on derivations of polynomial algebras where one can also find explicit generators of the kernels of concrete locally nilpotent derivations.

The following exercise is a partial case of a result proved by several authors for polynomial algebras (van den Essen [101], Shannon and Sweedler [235] and Abhyankar and Li [2]) and by Drensky, Gutierrez and Yu [90] in the noncommutative case.

Exercise 10.4.13 Let $F_{2m}(R)$ be a relatively free algebra with generators $x_1,\dots,x_m$, $y_1,\dots,y_m$ and let

$$\phi=(f_1(X),\dots,f_m(X)),\ \psi=(g_1(X),\dots,g_m(X))$$

be endomorphisms of the subalgebra $F_m(R)$ generated by $x_1,\dots,x_m$. Then ϕ and ψ are automorphisms of $F_m(R)$ with $\psi=\phi^{-1}$ if and only if the ideals U and V of $F_{2m}(R)$ coincide, where

$$U=(x_i-f_i(Y)\mid i=1,\dots,m),\ V=(y_i-g_i(X)\mid i=1,\dots,m).$$

Hint. Work modulo the ideals U and V. If ϕ and ψ are automorphisms and $\psi=\phi^{-1}$, then use that

$$f_i(g_1(X),\dots,g_m(X))=\psi\circ\phi(x_i)=x_i,\ i=1,\dots,m,$$

and derive that $U\subseteq V$. If $U=V$, use that in $\overline{F_{2m}(R)}=F_{2m}(R)/U=F_{2m}(R)/V$

$$\bar{x}_i=f_i(\bar{y}_1,\dots,\bar{y}_m)=f_i(g_1(\bar{X}),\dots,g_m(\bar{X}))$$

and $\overline{\psi\circ\phi(x_i)}=\bar{x}_i$, i.e. the composition $\psi\circ\phi$ induces the identity map on $\bar{F}_m(R)$. Besides, the subalgebra generated by $\bar{x}_1,\dots,\bar{x}_m$ is isomorphic to $F_m(R)$. Hence $\mathrm{Ker}\phi=(0)$, $\mathrm{Im}\psi=F_m(R)$. Similarly, $\mathrm{Im}\phi=F_m(R)$ and ϕ is an automorphism. If difficulties appear, see the paper of Drensky, Gutierrez and Yu [90].

Remark 10.4.14 In the case of the polynomial algebra $K[V_m]$, if

$$\phi = (f_1(X), \dots, f_m(X))$$

is an endomorphism of $K[V_m]$, there exists a good algorithm based on Gröbner bases techniques, which allows to solve effectively the problem whether the ideal

$$(x_1 - f_1(Y), \dots, x_m - f_m(Y)) \triangleleft K[x_1, \dots, x_m, y_1, \dots, y_m]$$

has a generating set of the form

$$(y_1 - g_1(X), \dots, y_m - g_m(X)),$$

i.e. to determine whether ϕ is an automorphism and to find its inverse. For details see some book on Gröbner bases (e.g. by Adams and Loustaunau [4]) or the papers of van den Essen [101], Shannon and Sweedler [235] or Abhyankar and Li [2].

We have already seen in Exercise 10.2.9 that the Jacobian matrix of an automorphism of the polynomial algebra $K[V_m]$ is invertible. On the other hand, the inverse function theorem in calculus states that if the Jacobian matrix of a mapping $\mathbb{R}^m \to \mathbb{R}^m$ is invertible, then the mapping is *locally* invertible. The analogue of this theorem for the polynomial algebra gives that if the Jacobian matrix of an augmentation preserving endomorphism is invertible, then the endomorphism induces an automorphism of the algebra of the formal power series. The famous Jacobian conjecture is the following.

Jacobian Conjecture 10.4.15 If $\mathrm{char} K = 0$, then an endomorphism ϕ of the polynomial algebra $K[V_m] = K[x_1, \dots, x_m]$ is an automorphism if and only if the Jacobian matrix $J(\phi)$ of ϕ is invertible.

A lot of work has been done on the problem (see e.g. the book by Abhyankar [1] and the survey article of Bass, Connell and Wright [24]) but the answer is still unknown even for $m = 2$.

Exercise 10.4.16 Show that the condition $\mathrm{char} K = 0$ in the Jacobian conjecture is essential.

Hint. Show that for p prime, the Jacobian matrix of $\phi \in \mathrm{End}\mathbb{Z}_p[x]$, where $\phi(x) = x + x^p$, is invertible and ϕ is not an automorphism.

Remark 10.4.17 There are many generalizations of the partial derivatives $\partial/\partial x_i$ in the case of free and relatively free algebras and one can define the Jacobian matrix of $\phi \in \mathrm{End} K\langle V_m\rangle$ and $\phi \in \mathrm{End} F_m(R)$. Usually this Jacobian matrix satisfies the chain rule of Exercise 10.2.9 and one may ask an analogue of the Jacobian conjecture for free and relatively free algebras. There are also some other analogues of the Jacobian matrix, which are endomorphisms and not matrices. For details for $K\langle V_m\rangle$ see the paper of Dicks and Lewin [59] and the book by Schofield [234]. For an exposition on the general case see the survey article by Drensky [81].

10.5 Automorphisms of Relatively Free Algebras

Till the end of the chapter we shall shortly discuss automorphisms of relatively free algebras. We start with the theorem of Bergman [34] which gives a new class of automorphisms of relatively free algebras and shows that important relatively free algebras of rank 2 have wild automorphisms.

Theorem 10.5.1 (Bergman [34]) *Let* $F_2(M_k(K))$ *be the relatively free algebra of rank 2 in the variety generated by the* $k \times k$ *matrix algebra,* $k > 1$ *(i.e.* $F_2(M_k(K))$ *is the algebra of two generic* $k \times k$ *matrices). Let* $u(x,y) \neq 0$ *be a polynomial without constant term, in the centre of* $F_2(M_k(K))$ *and such that all variables participate in commutators only (i.e.* $u(x,y) \in B_2/(T(M_k(K)) \cap B_2)$ *in the notation of Definition 4.3.1). Then the endomorphism of* $F_2(M_k(K))$

$$\phi_u = (x + u(x,y),\ y)$$

is a wild automorphism.

Proof. We consider the canonical algebra homomorphisms

$$K\langle x,y\rangle \longrightarrow F_2(M_k(K)) \longrightarrow K[x,y]$$

which induce group homomorphisms

$$\mathrm{Aut}K\langle x,y\rangle \longrightarrow \mathrm{Aut}F_2(M_k(K)) \longrightarrow \mathrm{Aut}K[x,y].$$

By Corollary 10.3.8, the homomorphism

$$\mathrm{Aut}K\langle x,y\rangle \longrightarrow \mathrm{Aut}K[x,y]$$

is an isomorphism. This implies that

$$\mathrm{Aut}K\langle x,y\rangle \longrightarrow \mathrm{Aut}F_2(M_k(K))$$

is a monomorphism and

$$\mathrm{Aut}F_2(M_k(K)) \longrightarrow \mathrm{Aut}K[x,y]$$

is an epimorphism. We shall prove the theorem if we find a nontrivial automorphism of $F_2(M_2(K))$, and it will be ϕ_u, which is in the kernel of the homomorphism

$$\mathrm{Aut}F_2(M_k(K)) \longrightarrow \mathrm{Aut}K[x,y].$$

Let $v(x,y)$ be another element of $F_2(M_k(K))$ with the same properties as u. Since all variables participate in $v(x,y)$ in commutators only, and u is a central element in $F_2(M_k(K))$, we obtain that

$$v(x+u(x,y),y)=v(x,y),$$

$$\phi_u\circ\phi_v(x)=\phi_u(x+v(x,y))=\phi_u(x)+v(\phi_u(x),\phi_u(y))=$$

$$=x+u(x,y)+v(x+u(x,y),y)=x+u(x,y)+v(x,y)=\phi_{u+v}(x),$$

$$\phi_u\circ\phi_v(y)=\phi_u(y)=y=\phi_{u+v}(y)$$

and $\phi_u\circ\phi_v=\phi_{u+v}$. Hence ϕ_u is invertible with $\phi_u^{-1}=\phi_{(-u)}$. Clearly, ϕ_u induces on $K[x,y]$ the identity map and this completes the proof of the theorem.

Example 10.5.2 Let $k=2$ and let $u=[x,y]^2$. We know that u is in the centre of the algebra of two generic 2×2 matrices (see Exercise 7.3.2). By Bergman Theorem 10.5.1,

$$\phi_u=(x+[x,y]^2,\ y)$$

is a wild automorphism.

Remark 10.5.3 The algebra of two generic 2×2 matrices has an important property which does not hold for other algebras of generic matrices. We refer to the book by Formanek [108] or to the paper by Alev and Le Bruyn [9]. We shall only mention the following. If x and y are two generic 2×2 matrices and we consider the commutative algebra C generated by

$$\mathrm{tr}(x),\ \mathrm{tr}(y),\ \det(x),\ \det(y),\ \mathrm{tr}(xy),$$

then the algebra C is isomorphic to the polynomial algebra in five commuting variables. The automorphisms ϕ_u in Theorem 10.5.1 induce automorphisms of C and there was a hope that some "strange" automorphisms of the generic matrix algebra will induce "strange" automorphisms of the polynomial algebra. It turns out that in this way one can induce for example the Nagata automorphism in Exercise 10.4.4 (but of the algebra of 5 variables, where it is tame) and many other interseting automorphisms. See also the paper by Drensky and Gupta [89].

Remark 10.5.4 Using the idea of the proof of Theorem 10.5.1 combined with structure theory of relatively free algebras, one can show that for any noncommutative PI-algebra R, the relatively free algebra $F_2(R)$ has wild automorphisms, see the paper by Drensky [79]. There are also other methods for constructing wild automorphisms of relatively free algebras, see e.g. the survey article [81].

11. Free Lie Algebras and Their Automorphisms

There are many books on the theory of free Lie algebras, as the books by Bourbaki [37], Bahturin [21], Reutenauer [228] and the recent book by Mikhalev and Zolotykh [184] on free *Lie superalgebras* (with a long list of references, including very recent results), etc. In this chapter we first survey some results on free Lie algebras which show that their combinatorics is very different from the combinatorics of free associative algebras. The proofs can be found e.g. in the books by Bahturin [21] or Mikhalev and Zolotykh [184].

Comparing the results on automorphisms of free algebras in the case of commutative, associative and Lie algebras, and on automorphisms of free groups, the picture is much better in the case of groups and Lie algebras. We show how the combinatorial results on free Lie algebras imply the tameness of their groups of automorphisms. We also state and prove the Jacobian conjecture for free Lie algebras. Finally, we deal with automorphisms of the free metabelian Lie algebra of rank 2 and automorphisms of relatively free nilpotent Lie algebras.

11.1 Bases and Subalgebras of Free Lie Algebras

We assume that K is any field. We fix an integer $m > 1$ and denote by $K\langle V_m\rangle$ the free associative algebra freely generated by $x_1, \ldots, x_m$ (and $V_m = \text{span}\{x_1, \ldots, x_m\}$). By Witt Theorem 1.3.5, the free Lie algebra L_m is isomorphic to the Lie subalgebra generated by $x_1, \ldots, x_m$ in $K\langle V_m\rangle$. Hence, every element of L_m is a linear combination of associative monomials (words) in $x_1, \ldots, x_m$. In the literature, there are different bases of L_m as a vector space, e.g. these of Hall [123], Lyndon [174] and Shirshov [239]. We give without proof the description of the basis of Lyndon-Shirshov in the version of Shirshov.

We introduce the lexicographic ordering on the set of all nonempty words in $x_1, \ldots, x_m$ assuming that

$$u = x_{i_1} \ldots x_{i_p} \prec x_{j_1} \ldots x_{j_q} = v$$

if $i_1 = j_1, \ldots, i_k = j_k$ for some $k \geq 0$ and $i_{k+1} < j_{k+1}$. Additionally, if $p > q$ and $i_1 = j_1, \ldots, i_q = j_q$ (i.e. v is a beginning of u), we also assume that $u \prec v$.

For a noncommutative word $[u]$ with some distribution of the Lie brackets (e.g. $[u] = [x_2, [x_1, x_2]]$), we denote by $u = \overline{[u]}$ the *associative carrier* of $[u]$, i.e. the associative word obtained by deleting the brackets (e.g. if $[u] = [x_2, [x_1, x_2]]$, then $u = \overline{[u]} = x_2x_1x_2$).

Definition 11.1.1 Let

$$[U] = \bigcup_{n\geq 1} [U_n]$$

be the set of Lie commutators in $K\langle V_m\rangle$ constructed inductively in the following way.

1. Let $[U_1] = \{x_1, \ldots, x_m\}$. (We write x_i instead of $[x_i]$.)

2. If we have already constructed $[U_k]$ for $k = 1, \ldots, n-1$, then $[U_n]$ consists of all commutators $[w]$ of degree n such that $[w] = [[u], [v]]$, where $[u] \in [U_k]$, $[v] \in [U_{n-k}]$ and $[u], [v], [w]$ satisfy the conditions

(i) $\overline{[u]} \succ \overline{[w]} \succ \overline{[v]}$;

(ii) If $[u] = [[u_1], [u_2]]$, then $\overline{[u_2]} \preceq \overline{[v]}$.

Example 11.1.2 Let $m = 2$ and $x \prec y$. Then

$$[U_1] = \{x, y\},\ [U_2] = \{[y, x]\}$$

and $y \succ yx \succ x$;

$$[U_3] = \{[[y, x], x], [y, [y, x]]\},$$

$$y \succ yyx \succ yx \succ yxx \succ x;$$

$$[U_4] = \{[y, [y, [y, x]]], [y, [[y, x], x]]], [[[y, x], x], x]\},$$

$$y \succ yyyx \succ yyx \succ yyxx \succ yx \succ yxx \succ yxxx \succ x.$$

Theorem 11.1.3 (Lyndon [174], Shirshov [239]) *The set $[U]$ constructed in Definition 11.1.1 is a basis of the free Lie algebra L_m. This basis is called the basis of Lyndon-Shirshov.*

For a polynomial $g \in L_m$ we denote by g' the homogeneous component of highest degree (with respect to the usual degree defined as the length of the monomials in $K\langle V_m\rangle$). We say that the nonzero elements $g_1, \ldots, g_k$ of L_m satisfy a *nontrivial relation*, if there exists a nonzero Lie polynomial

$$f(z_1, \ldots, z_k) \in L_k \subset K\langle z_1, \ldots, z_k\rangle,$$

such that $f(g_1, \ldots, g_k) = 0$.

Based on his combinatorics of words, Shirshov [237] proved the following result which is crucial in our considerations.

Proposition 11.1.4 (Shirshov [237]) *Let $g_1, \ldots, g_k$ be elements of L_m which satisfy some nontrivial relation. Then one of the homogeneous components* *of highest degree* $g_1',\dots,g_k'$ *belongs to the Lie algebra generated by the other components of highest degree.*

The proof of Proposition 11.1.4 is a consequence of the proof of [237, Theorem 2]. It may also be found in [184] in the more general setup of free Lie superalgebras. I believe that having the experience with the combinatorics of words in Chapter 9, the reader should be able to understand the proof e.g. in [184].

Example 11.1.5 Let

$$g_1=[[x,y,x],[x,y]]-[x,y],\ g_2=[x,y,x]+2[x,y],\ g_3=[x,y],$$

(these elements are not written as linear combinations of the basis elements of Lyndon-Shirshov). Then

$$[g_1+g_3,[g_2,g_3]]=0,$$

and the homogeneous components of highest degree

$$g_1'=[[x,y,x],[x,y]],\ g_2'=[x,y,x],\ g_3'=g_3=[x,y]$$

satisfy $g_1'=[g_2',g_3']$.

Using his Proposition 11.1.4, Shirshov [237] discovered an algorithm finding a minimal system of generators for a subalgebra of L_m generated by a given system of polynomials and proved the theorem for the freedom of the subalgebras of free Lie algebras.

Theorem 11.1.6 (Shirshov, [237]) *Let* $L(X)$ *be the free Lie algebra generated by a set* X*. Then every Lie subalgebra of* $L(X)$ *is free.*

Theorem 11.1.6 is a Lie analogue of the Nielsen-Schreier theorem which states that *every subgroup of the free group is free again.* Two different proofs of the Nielsen-Schreier theorem are given for example in [176] and [135].

Exercise 11.1.7 Prove the following partial case of Theorem 11.1.6. Every finite set of elements $\{g_1,\dots,g_k\}$ of the free Lie algebra L_m generates a free Lie subalgebra.

Hint. Let G be the subalgebra generated by $g_1,\dots,g_k$. If the leading homogeneous components $g_1',\dots,g_k'$ are *independent* (i.e. satisfy no relation), then by Proposition 11.1.4 the elements $g_1,\dots,g_k$ themselves generate a free Lie algebra. If $g_1',\dots,g_k'$ are dependent, then one of them, say g_1', belongs to the subalgebra generated by $g_2',\dots,g_k'$. Let $f(z_2,\dots,z_k)$ be an element of the free Lie algebra L_{k-1} generated by $z_2,\dots,z_k$ such that $g_1'=f(g_2',\dots,g_k')$. We define a grading on L_{k-1} by $\deg z_i=\deg g_i'$. Without loss of generality we may assume that $f(z_2,\dots,z_k)$ is homogeneous of degree $\deg g_1' = \deg g_1$. The elements

$$h_1 = g_1 - f(g_2,\dots,g_k), h_i = g_i,\, i = 2,\dots,k,$$

generate the same subalgebra G as $g_1,\dots,g_k$. If $h_1 = 0$, then we apply induction on the number of the generators of G. If $h_1 \neq 0$, then the degree of h_1 is smaller that the degree of g_1 and we apply induction on $\sum_{i=1}^{k} \deg g_i$.

11.2 Automorphisms of Free Lie Algebras

Cohn [50] described the automorphisms of the free Lie algebra L_m of any finite rank m. Now this result is considered as classical. As in the case of polynomial and free associative algebras, an automorphism of L_m is *tame* if it is a composition of linear and triangular automorphisms.

Theorem 11.2.1 (Cohn [50]) *For every integer $m \geq 2$ the automorphisms of the free Lie algebra L_m are tame.*

Idea of the Proof. The original proof of Cohn [50] is based on the generalization of the Euclidean algorithm to the noncommutative case and on the technique of free ideal rings. Another proof based on the Shirshov technique in free Lie algebras is given by Kukin [158]. Here we sketch his proof. It repeats the main steps of the hint of Exercise 11.1.7. Let $\phi \in \mathrm{Aut} L_m$ and let $g_i = \phi(x_i)$, $i = 1,\dots,m$. Clearly, $\{g_1,\dots,g_m\}$ is a system of free generators of L_m. We apply induction on $\sum_{i=1}^{m} \deg g_i$. If this sum is equal to m, then all g_i are linear combinations of $x_1,\dots,x_m$. Since ϕ is an automorphism, we obtain that ϕ is an invertible linear transformation of V_m, and ϕ is a tame automorphism. Let $\sum_{i=1}^{m} \deg g_i > m$. Since $g_1,\dots,g_m$ generate L_m, there exists Lie polynomials $f_j(z_1,\dots,z_m)$ without linear terms and constants $\alpha_{ij} \in K$, $i,j = 1,\dots,m$, such that

$$x_j = \sum_{i=1}^{m} \alpha_{ij} g_i + f_j(g_1,\dots,g_m),\ j = 1,\dots,m.$$

If all $f_j(z_1,\dots,z_m)$ are equal to 0, then the leading homogeneous components $g_1',\dots,g_m'$ are linearly dependent and one of them is a Lie polynomial (of first degree) of the others. If some f_j is nonzero, then the leading homogeneous components $g_1',\dots,g_m'$ are dependent because in the left hand side of the above equations there are monomials of first degree only. Again, $g_1',\dots,g_m'$ are dependent and, by Proposition 11.1.4, one of them is a polynomial of the others. If $g_1' = f(g_2',\dots,g_m')$, then, as in Exercise 11.1.7, the elements

$$h_1 = g_1 - f(g_2,\dots,g_m),\ h_i = g_i,\ i = 2,\dots,m,$$

generate L_m and

$$\sum_{i=1}^{m} \mathrm{deg} h_i < \sum_{i=1}^{m} \mathrm{deg} g_i.$$

The mapping $g_i \to h_i$, $i = 1, \dots, m$, induces a triangular automorphism τ of L_m (with respect to the system of free generators $\{g_1, \dots, g_m\}$ of L_m). Hence, the composition $\psi = \tau \circ \phi$ is an automorphism of L_m and

$$\psi(x_i) = \tau(\phi(x_i)) = \tau(g_i) = h_i,\ i = 1, \dots, m.$$

By induction, ψ is tame and this implies that ϕ is tame. With some additional considerations, this proof gives also an algorithm how to decompose an automorphism into a product of linear and triangular automorphisms.

Remark 11.2.2 Since the only Lie elements in one variable x are its scalar multiples αx, $\alpha \in K$, for $m = 2$ the Cohn Theorem 11.2.1 gives that

$$\mathrm{Aut} L_2 \cong GL(V_2) \cong GL_2(K).$$

We continue our exposition with the Lie algebra analogue of Jacobian Conjecture 10.4.15. Such considerations were first carried out for free groups by Birman [35] who solved the *Jacobian conjecture for free groups.* The further development of the approach of Birman is very useful also for free and relatively free associative algebras. It helps to show that some classes of endomorphisms are not automorphisms and to construct wild automorphisms of relatively free algebras. First we introduce partial derivatives.

Definition 11.2.3 (i) Let $f(x_1, \dots, x_m)$ be an element of the free associative algebra $K\langle V_m\rangle = K\langle x_1, \dots, x_m\rangle$. Write f in the form

$$f = f(x_1, \dots, x_m) = \alpha + \sum_{i=1}^{m} x_i f_i(x_1, \dots, x_m),\ \alpha \in K, f_i \in K\langle V_m\rangle.$$

We call the polynomials

$$\frac{\partial f}{\partial x_i} = \frac{\partial_r f}{\partial_r x_i} = f_i(x_1, \dots, x_m),\ i = 1, \dots, m,$$

the (*right*) *partial* (*Fox*) *derivatives* of f.

(ii) If ϕ is an endomorphism of $K\langle V_m\rangle$, the (*right*) *Jacobian matrix* of ϕ is the $m \times m$ matrix with entries from $K\langle V_m\rangle$

$$J(\phi) = J_r(\phi) = \left(\frac{\partial_r \phi(x_j)}{\partial_r x_i}\right).$$

Similarly we may introduce left partial Fox derivatives $\partial_l / \partial_l x_i$ and the left Jacobian matrix $J_l(\phi)$ of an endomorphism ϕ.

Exercise 11.2.4 Show that the Jacobian matrix in Definition 11.2.3 satisfies the chain rule

$$J_r(\phi\circ\psi)=J_r(\phi)\phi(J_r(\psi)),\ J_l^t(\phi\circ\psi)=\phi(J_r^t(\psi))J_l^t(\phi),\ \phi,\psi\in \mathrm{End}K\langle V_m\rangle,$$

where $\phi(J(\psi))$ means that ϕ acts on the entries of the matrix and ^{t} states for the transpose matrix. Derive from here that if ϕ is an automorphism, then its right Jacobian matrix has an inverse from the right.

Hint. Let e.g. $m=2$ and $\phi=(f,g)$, $\psi=(p,q)$. Show that

$$\frac{\partial_r p(f,g)}{\partial_r x}=\frac{\partial_r f}{\partial_r x}\frac{\partial_r p}{\partial_r f}+\frac{\partial_r g}{\partial_r x}\frac{\partial_r p}{\partial_r g},\ \frac{\partial_l p(f,g)}{\partial_l x}=\frac{\partial_l p}{\partial_l f}\frac{\partial_l f}{\partial_l x}+\frac{\partial_l p}{\partial_l g}\frac{\partial_l g}{\partial_l x},$$

and similarly for the other derivatives.

Definition 11.2.5 Since we assume that the free Lie algebra L_m is a Lie subalgebra of the free associative algebra $K\langle V_m\rangle$, and every endomorphism ϕ of L_m is determined by the images of $x_1,\ldots,x_m$, we obtain that ϕ induces an endomorphism $\tilde\phi$ of $K\langle V_m\rangle$. We define the *Jacobian matrix* $J(\phi)$ of ϕ as

$$J(\phi)=J(\tilde\phi).$$

Exercise 11.2.6 (Reutenauer [227]) Let $m=2$ and let $\phi\in\mathrm{End}K\langle x,y\rangle$ be defined by

$$\phi(x)=x+yxy,\ \phi(y)=y.$$

Show that both the left and the right Jacobian matrices of ϕ are invertible but ϕ is not an automorphism of $K\langle x,y\rangle$.

Hint. Use, for example, the commutator test of Theorem 10.3.9 to see that ϕ is not an automorphism:

$$[\phi(x),\phi(y)]=[x,y]+yxy^2-y^2xy\neq\alpha[x,y],\ \alpha\in K.$$

On the other hand,

$$J_r(\phi)=\begin{pmatrix}1&0\\ xy&1\end{pmatrix},\ J_r^{-1}(\phi)=\begin{pmatrix}1&0\\ -xy&1\end{pmatrix}.$$

Exercise 11.2.6 shows that the Jacobian matrix $J(\phi)$ of Definition 11.2.3 does not carry enough information about the endomorphism ϕ of $K\langle V_m\rangle$ even for $m=2$. The "correct" definition of $J(\phi)$ for $\phi\in\mathrm{End}K\langle V_m\rangle$ is this of [59] and [234], see Remark 10.4.17. Nevertheless the following theorem of Reutenauer-Shpilrain-Umirbaev gives that for free Lie algebras the situation is completely another and our definition of the Jacobian matrix of $\phi\in\mathrm{End}L_m$ is sufficient for our study of the automorphisms of L_m. In particular, the theorem gives the affirmative answer to the Lie analogue of the Jacobian conjecture.

Theorem 11.2.7 (Reutenauer [227], Shpilrain [241], Umirbaev [254]) *Let* ϕ *be an endomorphism of the free Lie algebra* L_m. *The right Jacobian matrix of* ϕ *is invertible from the right (as a matrix with entries from* $K\langle V_m\rangle$*) if and only if* ϕ *is an automorphism.*

Proof. We follow the arguments of Umirbaev. If $\phi \in \mathrm{Aut}L_m$, then, by the chain rule of Exercise 11.2.4,

$$e = J_r(\phi \circ \phi^{-1}) = J(\phi)\phi(J(\phi^{-1})),$$

where e is the unity $m \times m$ matrix. In this way we obtain the "easy" part of the theorem. Now, let $J(\phi) = J_r(\phi)$ be invertible over $K\langle V_m\rangle$ and let $J_r(\phi)a = e$ for some $m \times m$ matrix a with entries from $K\langle V_m\rangle$. By Definition 11.2.3 of Fox derivatives, and since $f_j = \phi(x_j)$, $j = 1, \ldots, m$, are polynomials without constant terms, we obtain that

$$f_j = x_1\frac{\partial f_j}{\partial x_1} + \ldots + x_m\frac{\partial f_j}{\partial x_m},\ j = 1, \ldots, m,$$

and, in matrix form,

$$(f_1, \ldots, f_m)a = (x_1, \ldots, x_m)J(\phi)a = (x_1, \ldots, x_m).$$

Hence $x_1, \ldots, x_m$ belong to the right ideal generated by $f_1, \ldots, f_m$ in the universal enveloping algebra $U(L_m) = K\langle V_m\rangle$. Obviously $f_1, \ldots, f_m$ belong to the right ideal generated by $x_1, \ldots, x_m$, i.e. these two right ideals of $U(L_m)$ coincide. By Proposition 1.3.8, the Lie subalgebra of L_m generated by $f_1, \ldots, f_m$ coincides with the whole L_m and the endomorphism ϕ is an epimorphism. As in the case of polynomial and free associative algebras (see Exercise 10.1.11), this implies that ϕ is an automorphism.

Comparing the final form of the results on automorphisms of free groups and free Lie algebras with the partial results for polynomial and free associative algebras it seems to us that the main difference between groups and Lie algebras on the one hand and polynomial and free associative algebras on the other lies in the good combinatorics in the group and Lie cases. For example, the subgroups of the free groups are free again (the Nielsen-Schreier theorem) and a similar statement holds for the subalgebras of the free Lie algebras (Theorem 11.1.6 of Shirshov). There exist algorithms which, for any set of generators of the free group and the free Lie algebra, give a minimal system of free generators and transform this system to the canonical system of free generators $\{x_1, \ldots, x_m\}$ (see [176] for groups and [237] and [158] for Lie algebras). Each step of the algorithms corresponds to a tame automorphism. Unfortunately, these arguments do not work in the associative case.

11.3 Automorphisms of Relatively Free Lie Algebras

Very often the investigations on relatively free Lie algebras and, in particular, on their automorphisms, are inspired by similar investigations both in group theory and theory of associative PI-algebras. Maybe this is in the very nature of the relatively free Lie algebras. Some of them, as the free solvable or, more generally, the free polynilpotent Lie algebras, are direct analogues of the corresponding group objects. Sometimes the study of the group requires the study of the Lie algebra (as a "linearization" of the group problem). Sometimes the group theoretic result serves as a model or as a conjecture for the Lie algebra result. On the other hand, combinatorial methods developed for associative algebras often work also for Lie algebras. The results of the present section are a combination of both associative algebra and group trends in the study of automorphisms of relatively free Lie algebras.

As in the case of relatively free associative algebras (see the proof of Theorem 10.5.1), Remark 11.2.2 gives possibilities to construct wild automorphisms for relatively free Lie algebras of rank 2.

Exercise 11.3.1 Let L_2/L_2'' be the free metabelian Lie algebra generated by x and y. Let $0 \neq w \in L_2'/L_2''$. Show that the endomorphism $\exp(\mathrm{ad} w)$ defined by

$$\exp(\mathrm{ad} w)(u) = u + [u, w],\ u \in L_2/L_2'',$$

is an automorphism. Such an automorphism $\exp(\mathrm{ad} w)$ is called *inner*. Show that the inner automorphisms act identically on L_2'/L_2''.

Hint. Let $\phi = \exp(\mathrm{ad} w)$. By Exercise 3.2.2, $\mathrm{ad} w$ is a derivation of L_2/L_2'' and, since $w \in L_2'/L_2''$, we obtain that $\mathrm{ad}^2 w = 0$. Hence $\mathrm{ad} w$ is nilpotent and by Exercise 10.4.1, $\phi = \exp(\mathrm{ad} w)$ is an automorphism. Clearly, if $u \in L_2'/L_2''$, then $[u, w] = 0$ and $\phi(u) = u$. Pay attention that the result does not depend on the characteristic of the field K, because $\mathrm{ad}^2 w = 0$, the only denominator in the expression of $\exp(\mathrm{ad} w)$ is $1!$ in $\mathrm{ad} w/1!$ and the exponent is well defined.

The exercise can be also handled directly, without using Exercises 3.2.2 and 10.4.1. Verify that $\phi([u, v]) = [\phi(u), \phi(v)]$, $u, v \in L_2/L_2''$. For this purpose, using the anticommutativity, the Jacobi identity, and the metabelian identities $[[u, w], [v, w]] = 0$ and $[[u, v], w] = 0$ (since $w \in L_2'/L_2''$) show that

$$[\phi(u), \phi(v)] = [u + [u, w], v + [v, w]] =$$

$$= [u, v] + [u, [v, w]] + [[u, w], v] = [u, v] = \phi([u, v]),$$

i.e. ϕ is an automorphism of L_2/L_2'' which acts identically on L_2'/L_2''.

The above Exercise 11.3.1 is a partial case of a result of Shmelkin [240] (based on techniques coming from group theory), that the automorphism of the free solvable Lie algebra $L_m/L_m^{(s)}$ is inner if and only if it acts identically on $L_m^{(s-1)}/L_m^{(s)}$.

Exercise 11.3.2 Show that every nonidentical inner automorphism of the free metabelian Lie algebra of rank 2 is wild.

Hint. Use the same idea as in the proof of Bergman Theorem 10.5.1. Let L_2/L_2' be the two-dimensional abelian Lie algebra (i.e. a two-dimensional vector space with trivial multiplication and automorphism group isomorphic to $GL_2(K)$). Show that the Lie algebra homomorphisms

$$L_2 \longrightarrow L_2/L_2'' \longrightarrow L_2/L_2'$$

induce group homomorphisms

$$GL_2(K) = \mathrm{Aut}L_2 \longrightarrow \mathrm{Aut}L_2/L_2'' \longrightarrow \mathrm{Aut}L_2/L_2' = GL_2(K).$$

Use that every nonidentical automorphism of L_2/L_2'' which induces the identity map on L_2/L_2', is wild.

There is a general theorem of Bahturin and Nabiyev [22], which states that *for every $m \geq 2$, the inner automorphisms of the free metabelian Lie algebra L_m/L_m'' are wild.*

Exercise 11.3.3 Let $F_2(sl_2(K)) = L_2/(L_2 \cap T(sl_2(K))$ be the relatively free algebra of the variety of Lie algebras generated by the Lie algebra $sl_2(K)$ of traceless 2×2 matrices over a field of characteristic 0. Show that the endomorphism ϕ of $F_2(sl_2(K))$ defined by

$$\phi(x) = x + [[x,y,y],[x,y]],\ \phi(y) = y,$$

is a wild automorphism.

Hint. Show that $[[[x,y,y],[x,y]],y] = 0$ is a polynomial identity for $sl_2(K)$ and, as in Theorem 10.5.1, the inverse of ϕ is ψ, where

$$\psi(x) = x - [[x,y,y],[x,y]],\ \psi(y) = y.$$

Now we shall describe the automorphism group of the free metabelian Lie algebra of rank 2. The result follows from a theorem of Shmelkin [240] and is stated in this form in [79]. For further study of $\mathrm{Aut}L_2/L_2''$ involving also methods from representation theory of $GL_2(K)$, see the recent paper by Papistas [204].

Theorem 11.3.4 *Every automorphism of the free metabelian Lie algebra L_2/L_2'' of rank 2 is a composition of a linear and an inner automorphism.*

Proof. Let ϕ be an automorphism of L_2/L_2'' and let

$$\phi(x) = \alpha x + \beta y + f(x,y),\ \phi(y) = \gamma x + \delta y + g(x,y),$$

where $\alpha, \beta, \gamma, \delta \in K$ and $f, g \in L_2'/L_2''$. Since ϕ induces an automorphism of the two-dimensional abelian Lie algebra L_2/L_2', we obtain that $\phi = \psi \circ \tau$, where τ is the linear automorphism

$$\tau(x) = \alpha x + \beta y,\ \tau(y) = \gamma x + \delta y$$

and ψ induces the identity map modulo the commutator ideal. Hence, without loss of generality we may assume that

$$\phi(x) = x + f(x,y),\ \phi(y) = y + g(x,y),\ f, g \in L_2'/L_2'',$$

and we have to prove that ϕ is inner. The commutator ideal L_2'/L_2'' has a basis

$$\{[x,y](\mathrm{ad}^p x)(\mathrm{ad}^q y) \mid p, q \geq 0\}.$$

As in Chapter 6 (see the proof of Theorem 6.2.1) we define an action on L_2'/L_2'' of the polynomial algebra $K[t,u]$ in two commuting variables t and u by

$$t^r u^s * [x,y](\mathrm{ad}^p x)(\mathrm{ad}^q y) = [x,y](\mathrm{ad}^{p+r} x)(\mathrm{ad}^{q+s} y).$$

We write the images of x and y under ϕ and ϕ^{-1} as

$$\phi(x) = x + a * [x,y],\ \phi(y) = y + b * [x,y],$$

$$\phi^{-1}(x) = x + c * [x,y],\ \phi^{-1}(y) = y + d * [x,y],$$

where $a, b, c, d \in K[t,u]$. Let for example $a \neq 0$. Direct calculations show that

$$x = (\phi \circ \phi^{-1})(x) = \phi(x + c * [x,y]) = x + a * [x,y] + c * [x + a * [x,y], y + b * [x,y]] =$$

$$= x + a * [x,y] + c * ([x,y] + a * [[x,y],y] - b * [[x,y],x]) =$$

$$= x + (a + c(1 + au - bt)) * [x,y].$$

Since the action of $K[t,u]$ on L_2'/L_2'' is faithful, we obtain that

$$a + c(1 + au - bt) = 0.$$

Similarly, we derive from $x = (\phi^{-1} \circ \phi)(x)$

$$c + a(1 + cu - dt) = 0.$$

The above two equations give that

$$c = -a(1 + cu - dt),\ a = -c(1 + au - bt) = a(1 + au - bt)(1 + cu - dt).$$

Since $K[t,u]$ has no zero divisors and $a \neq 0$, we obtain that

$$(1 + au - bt)(1 + cu - dt) = 1$$

and this implies $au = bt$. Hence $a = v(t,u)t$, $b = v(t,u)u$ for some $v(t,u) \in K[t,u]$ and this means that

$$\phi(x) = x + v(t,u) * (t * [x,y]) = x - [x, v * [x,y]] = \exp(\mathrm{ad}(-v * [x,y])(x),$$

$$\phi(y) = y + v(t,u) * (u * [x,y]) = y - [y, v * [x,y]] = \exp(\mathrm{ad}(-v * [x,y])(y),$$

i.e. ϕ is an inner automorphism.

It is interesting to compare the structure of the automorphism groups of the free metabelian group G_m/G_m'' (where G_m is the free group of rank m) and the free metabelian Lie algebra L_m/L_m''. The results of Chein [47], Bachmuth and Mochizuki [17, 18, 19] and Romankov [229] give that $\mathrm{Aut}G_m/G_m''$ consists of tame automorphisms for all $m \neq 3$ and $\mathrm{Aut}G_3/G_3''$ is not finitely generated. Hence, by Theorem 10.1.2, G_3/G_3'' has a lot of wild automorphisms. On the other hand, the only wild automorphisms of $\mathrm{Aut}L_m/L_m''$ arise from the result of Bahturin and Nabiyev [22] for the wildness of the inner automorphisms of L_m/L_m''. The group analogues of the inner automorphisms are the automorphisms by conjugation and they are tame for any relatively free group. Freely restated, the following problem asks "how wild" are the automorphisms of L_m/L_m''.

Problem 11.3.5 Find generators of the automorphism group of the free metabelian Lie algebra L_m/L_m'' for $m > 2$. Is $\mathrm{Aut}L_m/L_m''$ generated by the tame and the inner automorphisms?

For $m \geq 4$ and $\mathrm{char}K = 0$, the tame automorphisms form quite a large subgroup of $\mathrm{Aut}L_m/L_m''$, see the paper by Bryant and Drensky [42]. (One needs some knowledge of the methods of the next Chapter 12 to read this paper.)

Exercise 11.3.6 Prove the analogue of Theorem 11.3.4 for the free associative "metabelian" algebra $K\langle x,y\rangle/C^2$, where C is the commutator ideal of $K\langle x,y\rangle$: *Every automorphism of* $K\langle x,y\rangle/C^2$ *is a composition of a tame automorphism and an inner automorphism* $\exp(\mathrm{ad}w)$, *where* $w \in C/C^2$.

Hint. Since C/C^2 has a basis

$$\{x^a y^b ([x,y]\mathrm{ad}^c x\ \mathrm{ad}^d y) \mid a,b,c,d \geq 0\},$$

define an action of the polynomial algebra in four variables on C/C^2 as in Chapter 6 and repeat the considerations in the proof of Theorem 11.3.4, replacing the linear automorphism with an automorphism of the polynomial algebra $K\langle x,y\rangle/C$. If difficulties appear, see the paper of Drensky [79].

There is an analogue of the Jacobian matrix for free metabelian Lie and associative algebras of any rank $m \geq 2$. The Jacobian matrix is a matrix with entries from the polynomial algebra in m variables for the Lie algebra case and in $2m$ variables in the associative algebra case. Results of Umirbaev [255, 256] show that in both cases the invertibility of the Jacobian matrix implies the invertibility of the endomorphism, i.e. the Jacobian conjecture has a positive solution for metabelian algebras.

Finally, we show that the nilpotent relatively free algebras have a lot of wild automorphisms. For $\mathrm{char}K = 0$ the result is a partial case of a result of Drensky and C.K. Gupta [88] based on applications of the representation theory of groups. A direct proof for the general case is in the paper of Bryant and Drensky [41] and is based on a method developed by Bryant, Gupta, Levin and Mochizuki [43] for constructing wild automorphisms of free nilpotent groups.

The following exercise is a well known fact for nilpotent algebras. Properly restated, it holds also for nilpotent groups.

Exercise 11.3.7 Let G be a finitely generated nilpotent (not necessarily Lie or associative) K-algebra. Show that every endomorphism ϕ of G which induces an invertible linear transformation on the vector space G/G^2, is an automorphism.

Hint. Show that ϕ induces invertible linear transformations ϕ_k on the factors G^k/G^{k+1}. If $g_1, \ldots, g_k \in G$, find elements $h_1, \ldots, h_k \in G$ such that $\phi(h_i) \equiv g_i \pmod{G^2}$; then $\phi(h_1 \ldots h_k) \equiv g_1 \ldots g_k \pmod{G^{k+1}}$. Hence ϕ_k is onto G^k/G^{k+1}. Since G is finitely generated, the vector spaces G^k/G^{k+1} are finite dimensional, and the "onto" linear transformations are invertible. Use that $G^n = 0$ for some sufficiently large n, and derive that ϕ is a 1-1 correspondence.

Theorem 11.3.8 *Let $T(G)$ be the T-ideal of a nilpotent but nonabelian Lie algebra G. Then for every $m \geq 2$, the relatively free algebra $F_m(G) = L_m/(T(G) \cap L_m)$ has wild automorphisms.*

Proof. Since G is not abelian, $[x_1, x_2] \neq 0$ in the relatively free algebra $F_m(G)$. Additionally, G does not satisfy any polynomial identity of the form $[x_1, x_2] + f = 0$, where $f \in L^3$ and L is the free Lie algebra of countable rank. (This is obvious if the base field K is infinite because the T-ideal $T(G)$ is generated by its homogeneous elements. Over a finite field K one needs some additional, but not very difficult inductive arguments on the class of nilpotency c of G: In order to make the inductive step, we use that

$$[[x_1, x_2] + f, x_3, \ldots, x_c] = [x_1, x_2, x_3, \ldots, x_c] + [f, x_3, \ldots, x_c] = 0$$

is a polynomial identity for G and derive that $[x_1, \ldots, x_c]$ belongs to $T(G) + L^{c+1}$. Hence G is nilpotent of class c, which is a contradiction.) Since $F_m(G)$ is nilpotent, by Exercise 11.3.7 the endomorphism ϕ defined by

$$\phi(x_1) = x_1 + [x_1, x_2],\ \phi(x_i) = x_i,\ i = 2, \ldots, m,$$

is an automorphism. We shall show that this automorphism is wild. Let the automorphism ψ of L_m be a lifting of ϕ. Then

$$\psi(x_1) = x_1 + [x_1, x_2] + f_1,\ \psi(x_i) = x_i + f_i,\ i = 2, \ldots, m,$$

where $f_1, \ldots, f_m$ belong to L^3. Hence, modulo the ideal of $K\langle V_m\rangle$ of the polynomials without constant and linear terms, the right Jacobian matrix $J(\psi)$ of ψ is equal to

$$\begin{pmatrix} 1+x_2 & 0 & 0 & \ldots & 0 & 0 \\ -x_1 & 1 & 0 & \ldots & 0 & 0 \\ 0 & 0 & 1 & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & 1 & 0 \\ 0 & 0 & 0 & \ldots & 0 & 1 \end{pmatrix}.$$

Since ψ is an automorphism, the matrix $J(\psi)$ is invertible. Hence its image $\overline{J(\psi)}$ in the set of $m \times m$ matrices with commuting polynomial entries is also invertible and its determinant is a nonzero constant in the field. Concrete calculations show that $\det(\overline{J(\psi)})$ is equal to

$$1 + x_2 + \text{ terms of higher degree}$$

which shows that ψ cannot be an automorphism.

For further reading on automorphisms of relatively free Lie algebras see the survey article by the author [81].

12. The Method of Representation Theory

In this chapter we deal with the powerful method of representation theory of groups in the study of PI-algebras. To the best of our knowledge, the origins of this method can be found in the papers by Malcev [180] and Specht [245] in 1950, where they applied for a first time representations of the symmetric group to the theory of PI-algebras. Then, starting in the 70's, the method was further developed, also in the language of representations of the symmetric group, in a series of papers by Regev (see e.g. his survey article [226]). Regev also involved other techniques, as asymptotic methods, combinatorics and even evaluations of multiple integrals, which nowadays are also considered as a part of the method of representation theory. In the early 80's, the author [71] and Berele [29] started to use the representation theory of the general linear group which had appeared incidentally in some papers by Razmyslov, Procesi and others. A lot of work was done also by Formanek (see [105]). Recently, one has started to apply also representations of Lie superalgebras, instead of representations of the general linear group (see Berele and Regev [32]).

The purpose of this chapter is to give some basic idea of the method with applications to the polynomial identities of the Grassmann algebra, the algebras of upper triangular matrices and the 2×2 matrices. We have also included some general facts about representation theory of groups and some results specific for the representations of the symmetric and the general linear groups. For more detailed background see for example the books by Lang [161], Kostrikin [153] or Curtis and Reiner [52]. A good source for knowledge on representation theory of symmetric and general linear groups are the books by Weyl [266], James and Kerber [129] or Macdonald [175]. For further reading on applications to PI-algebras we refer e.g. to the survey articles by Regev [226] and the author [77, 78]. It turns out that the technique under consideration is very useful also in the study of automorphisms of relatively free algebras. For exposition see the survey article [79] and the references there.

Although most of the considerations hold over any field K of characteristic 0, in this chapter we assume that K is an algebraically closed field of characteristic 0, e.g. $K = \mathbb{C}$. Sometimes we shall emphasize on the algebraic closeness of K if it is really essential for the considerations.

12.1 Representations of Finite Groups

We fix the notation G for a group. If G is finite, we denote by $|G|$ the number of its elements. If V is a vector space, we denote by $GL(V)$ the group of all invertible linear operators of V. If $\dim V = m < \infty$, fixing a basis of V, we identify the group $GL(V)$ with the group $GL_m = GL_m(K)$ of invertible $m \times m$ matrices with entries from K (see Exercise 1.1.4 (iii)).

Definition 12.1.1 Let $G = \{g \mid g \in G\}$ be a group. The K-algebra KG with basis as a vector space $\{g \mid g \in G\}$ and with multiplication between the basis vectors defined by the multiplication in G is called the *group algebra of* G (see also Exercise 1.1.4 (vii)).

Example 12.1.2 (i) Let $G = \{1, a, b, c\}$ be the elementary abelian 2-group, i.e. the abelian group generated by a, b with relations $a^2 = b^2 = 1$ (and $c = ab$). The group algebra KG is the four-dimensional vector space with basis $\{1, a, b, c\}$ and multiplication

$$(\varepsilon_1 + \alpha_1 a + \beta_1 b + \gamma_1 c)(\varepsilon_2 + \alpha_2 a + \beta_2 b + \gamma_2 c) =$$

$$= (\varepsilon_1\varepsilon_2 + \alpha_1\alpha_2 + \beta_1\beta_2 + \gamma_1\gamma_2) + (\varepsilon_1\alpha_2 + \alpha_1\varepsilon_2 + \beta_1\gamma_2 + \gamma_1\beta_2)a +$$

$$+ (\varepsilon_1\beta_2 + \beta_1\varepsilon_2 + \alpha_1\gamma_2 + \gamma_1\alpha_2)b + (\varepsilon_1\gamma_2 + \gamma_1\varepsilon_2 + \alpha_1\beta_2 + \beta_1\alpha_2)c.$$

(ii) If $G = \langle x \rangle$ is the infinite cyclic group generated by x, then the group algebra KG has a basis $\{1, x^{\pm 1}, x^{\pm 2}, \ldots\}$ and multiplication $x^k x^l = x^{k+l}$, $k, l \in \mathbb{Z}$. The algebra KG is isomorphic to the algebra of *Laurent polynomials in one variable.*

Definition 12.1.3 Let G be a group and let V be a vector space.

(i) A *representation* ϕ of G in V is a group homomorphism

$$\phi : G \longrightarrow GL(V).$$

The *degree* of the representation ϕ is equal to the dimension of the vector space V. The representation ϕ is *faithful* if the kernel of ϕ is trivial; ϕ is *trivial*, if its kernel coincides with G.

(ii) Two representations $\phi : G \longrightarrow GL(V)$ and $\phi' : G \longrightarrow GL(V')$ are called *equivalent* or *isomorphic*, if there exists an isomorphism $\theta : V \longrightarrow V'$ of the vector spaces V and V' such that

$$(\theta \circ \phi(g))(v) = (\phi'(g) \circ \theta)(v), \ v \in V, \ g \in G.$$

(iii) If W is a subspace of V such that $(\phi(G))(W) = W$, then the representation $\psi : G \longrightarrow GL(W)$ defined by

$$(\psi(g))(w) = (\phi(g))(w), \ g \in G, \ w \in W \subseteq V,$$

is called a *subrepresentation* of the representation $\phi : G \longrightarrow GL(V)$. The subrepresentation ψ is *proper* if $W \neq \{0\}$ and $W \neq V$.

(iv) If $\phi' : G \longrightarrow GL(V')$ and $\phi'' : G \longrightarrow GL(V'')$ are two representations of G, then the representation $\phi = \phi' \oplus \phi'' : G \longrightarrow GL(V' \oplus V'')$ defined by

$$(\phi(g))(v', v'') = ((\phi'(g))(v'), (\phi''(g))(v'')),\ g \in G,\ (v', v'') \in V' \oplus V'',$$

is called the *direct sum* of ϕ' and ϕ''. Similarly one defines the direct sum of any (finite or infinite) number of representations. The *tensor product* $\phi = \phi' \otimes \phi'' : G \longrightarrow GL(V' \otimes V'')$ of the representations ϕ' and ϕ'' is defined by

$$(\phi(g))(v' \otimes v'') = (\phi'(g))(v') \otimes (\phi''(g))(v''),\ g \in G,\ v' \otimes v'' \in V' \otimes V''.$$

(v) The representation $\phi : G \longrightarrow GL(V)$ is *irreducible* if it has no proper subrepresentations; ϕ is *completely reducible* if it is a direct sum of irreducible representations.

Remark 12.1.4 Since every representation $\phi : G \longrightarrow GL(V)$ of a group G in V defines a linear action of G on V, we may consider V as a left KG-module. Sometimes we say that V is a G-module instead a KG-module. The main notions as subrepresentations, irreducibility, etc. for ϕ correspond to the similar notions for the G-module V. In particular, the *regular representation* ρ of G is the representation of G in KG defined by

$$\rho(g) : \sum_{h \in G} \alpha_h h \to \sum_{h \in G} \alpha_h gh,\ g \in G,\ \alpha_h \in K.$$

Considering KG as a left G-module, we always assume that G acts on KG in this way, as a group of left translations. The subrepresentations of ρ correspond to left ideals of the group algebra KG and the irreducible subrepresentations correspond to the minimal left ideals of KG.

Exercise 12.1.5 Let $\phi : G \longrightarrow GL(V)$ be a representation of G in the m-dimensional vector space V. Show that ϕ is *decomposable* (i.e. has a proper subrepresentation) if and only if there exists a basis $\{v_1, \ldots, v_k, v_{k+1}, \ldots, v_m\}$ of V such that the matrices of $\phi(g)$, $g \in G$, with respect to this basis, have the block form

$$\phi(g) = \begin{pmatrix} a_g & 0 \\ b_g & c_g \end{pmatrix},$$

where a_g, b_g and c_g are respectively $k \times k$, $(m-k) \times k$ and $(m-k) \times (m-k)$ matrices. Prove that ϕ is *reducible*, i.e. a direct sum of proper subrepresentations if and only if the basis of V can be chosen in such a way that the matrices b_g have only zero entries for all $g \in G$.

Maschke Theorem 12.1.6 *Every finite dimensional representation of a finite group G is completely reducible. If the base field K is algebraically* *closed, then the group algebra KG is isomorphic to a direct sum of matrix algebras,*

$$KG \cong M_{d_1}(K) \oplus \ldots \oplus M_{d_r}(K).$$

Exercise 12.1.7 Let G be a finite group. Show that every irreducible G-module is isomorphic to a minimal left ideal of the group algebra KG.

Hint. If V is an irreducible G-module and v is a fixed nonzero element of V, then the mapping $\phi : KG \rightarrow V$, defined by

$$\phi : \sum_{g \in G} \alpha_g g \rightarrow \sum_{g \in G} \alpha_g g v, \ \alpha_g \in K,$$

is a homomorphism of G-modules (where the G-module KG corresponds to the regular representation of G). Since the kernel of ϕ is a submodule of KG, i.e. a left ideal of KG, by Maschke Theorem 12.1.6, there exists a G-submodule W of KG such that $KG = \mathrm{Ker}\phi \oplus W$ and $V \cong W$ as G-modules. Since V is irreducible, we obtain that W is a minimal left ideal of KG.

Exercise 12.1.8 Let G be a finite group and let

$$KG \cong M_{d_1}(K) \oplus \ldots \oplus M_{d_r}(K).$$

Show that, up to equivalence, the irreducible representations $\phi : G \longrightarrow GL(V)$ of G are the following. The G-module V is a d_i-dimensional vector space with the canonical action of the matrix algebra $M_{d_i}(K) \subseteq KG$ and the other matrix algebras $M_{d_j}(K)$, $j \neq i$, act trivially on V (sending the elements of V to 0). Derive from here that every finite group has a finite number of nonisomorphic irreducible representations.

Hint. Use Exercise 12.1.7 and the fact that the minimal left ideals of the matrix algebra $M_{d_i}(K)$ are d_i-dimensional vector spaces which are isomorphic as $M_{d_i}(K)$-modules. If difficulties appear, see for example, the book by Herstein [126].

Definition 12.1.9 Let $V_1, \ldots, V_r$ be all pairwise nonisomorphic modules of the finite group G. Let W be a finite dimensional G-module. Let $W = W_1 \oplus \ldots \oplus W_s$ be a *decomposition of* W *into a sum of irreducible* G*-submodules.* If m_i of the modules $W_1, \ldots, W_s$ are isomorphic to V_i, we call the nonnegative integer m_i the *multiplicity* of V_i in the decomposition of W and write

$$W = m_1 V_1 \oplus \ldots \oplus m_r V_r.$$

The following exercise is a restatement of the famous Schur Lemma (see e.g. Herstein [126] or Lang [161]).

Exercise 12.1.10 (i) Show that the centre of the matrix algebra $M_d(K)$ consists of all scalar matrices.

(ii) Let G be a finite group, let $\phi : V \longrightarrow W$ be a homomorphism of irreducible G-modules and let $0 \neq v \in V$. Show that either $\phi = 0$ or ϕ is an isomorphism and there exist a unique up to a multiplicative constant element $w \in W$ (which depends on v only) and a nonzero constant $\beta \in K$ (which depends on ϕ) such that

$$\phi\left(\sum_{g\in G} \alpha_g g v\right) = \beta \sum_{g\in G} \alpha_g g w,\ \alpha_g \in K.$$

Hint. (i) If $a = \sum_{i,j=1}^{d} \alpha_{ij} e_{ij}$, $\alpha_{ij} \in K$, is in the centre of $M_d(K)$, commuting a with matrix units e_{pq}, show that $\alpha_{ij} = 0$ if $i \neq j$ and $\alpha_{ii} = \alpha_{11}$.

(ii) Use that the kernel and the image of ϕ are submodules of V and W, respectively, and V and W are irreducible modules. Hence, either $\phi = 0$ and $\mathrm{Ker}\phi = V$ or $\mathrm{Ker}\phi = 0$, $\mathrm{Im}\phi = W$ and $V \cong W$. In the latter case, use Exercise 12.1.8. Assume that the vector spaces V and W coincide and $\{v_1 = v, v_2, \ldots, v_d\}$ is a basis of V. Consider ϕ as an invertible linear operator of V. The condition that ϕ is a G-module isomorphism implies that ϕ commutes with all linear operators of V, and ϕ is a scalar multiplication by (i).

Exercise 12.1.11 (i) Let I be a minimal left ideal of the matrix algebra $M_d(K)$. Show that there exists a nonzero element $i \in I$ such that $i^2 = i$. This element is called a *minimal idempotent* of I. Find the minimal idempotent when I consists of all matrices with nonzero entries in the first column only.

(ii) Show that for every minimal left ideal I of the group algebra KG of the finite group G there exists a minimal idempotent $e \in I$ such that $I = KG \cdot e$.

Hint. (i) Clearly $J = I \cdot M_d(K)$ is a two-sided ideal of $M_d(K)$. Since the matrix algebra is *simple*, i.e. has only trivial two-sided ideals, $J = M_d(K)$. If $I^2 = 0$, then

$$J^2 = (I(M_d(K) \cdot I))M_d(K) \subseteq I^2 \cdot M_d(K) = 0,$$

which is impossible. Hence $0 \neq I^2 \subseteq I$. Since I^2 is a left ideal, from the minimality of I we obtain that $I^2 = I$. If $0 \neq a \in I$, $Ia \neq 0$, then there exists an $i \in I$ such that $a = ia$ (why?) and, therefore $ia = i^2 a$, $(i^2 - i)a = 0$. The set $I_0 = \{j \in I \mid ja = 0\}$ is a left ideal of $M_d(K)$ and either $I_0 = 0$ or $I_0 = I$. Since $i \notin I_0$, we obtain that $I_0 = 0$ and $i^2 - i = 0$.

(ii) By Exercise 12.1.8, every minimal left ideal of KG is a minimal left ideal of a matrix subalgebra of KG and we apply (i). Since the minimal left ideal is generated by any of its nonzero elements, we obtain that $I = KG \cdot e$ for some minimal idempotent e.

Definition 12.1.12 Let $\phi : G \to GL(V)$ be a finite dimensional representation of the group G. The function $\chi_\phi : G \longrightarrow K$ defined by

$$\chi_\phi(g) = \mathrm{tr}_V(\phi(g)),\ g \in G,$$

is called the *character* of ϕ. If ϕ is an irreducible representation, then χ_ϕ is called an *irreducible character*.

Exercise 12.1.13 If g_1 and g_2 are two conjugate elements of the group G and ϕ is a finite dimensional representation of G, show that $\chi_\phi(g_1) = \chi_\phi(g_2)$, i.e. the characters are constant on the conjugacy classes of the group.

Hint. Use that the conjugate matrices have the same characteristic polynomials and hence the same traces.

Exercise 12.1.14 Let ϕ and ψ be two finite dimensional representations of the group G. Show that

$$\chi_{\phi\oplus\psi} = \chi_\phi + \chi_\psi,\ \chi_{\phi\otimes\psi} = \chi_\phi \cdot \chi_\psi.$$

Hint. Let V and W be the G-modules corresponding to the representations ϕ and ψ, respectively. Choose bases of V and W and construct bases of $V \oplus W$ and $V \otimes W$. Express the traces of $(\phi \oplus \psi)(g)$ and $(\phi \otimes \psi)(g)$ by the traces of $\phi(g)$ and $\psi(g)$ for $g \in G$.

The following theorem shows that the knowledge of the character gives a lot of information for the representation and the number of the irreducible representations (which is a ring theoretic property) is determined by a purely group property of the group. Usually, in the text books the second part of the theorem is derived as a consequence of the first part.

Theorem 12.1.15 *Let G be a finite group and let the field K be algebraically closed.*

(i) *Every finite dimensional representation of G is determined up to isomorphism by its character.*

(ii) *The number of the nonisomorphic irreducible representations of G is equal to the number of conjugacy classes of G.*

Remark 12.1.16 For groups G of small order one gives the *table of irreducible characters* of G. The rows of the table are labeled by the irreducible characters and the columns by representatives of the conjugacy classes. The entries of the table are equal to the values of the characters on the corresponding conjugacy classes.

Exercise 12.1.17 (i) Let $G = \langle g \mid g^n = 1\rangle$ be the cyclic group of order n and let ε be a primitive n-th root of 1. Show that the irreducible characters χ_j, $j = 0, 1, \dots, n-1$, of G are defined by

$$\chi_j(g^k) = \varepsilon^{jk},\ k = 0, 1, \dots, n-1.$$

(ii) If G is a direct product of the cyclic groups $\langle g \mid g^m = 1\rangle$ and $\langle h \mid h^n = 1\rangle$, show that all irreducible characters of G are

$$\chi_{pq}(g^k h^l) = \xi^{kp}\eta^{lq},\ p = 0, 1, \dots, m-1,\ q = 0, 1, \dots, n-1,$$

where ξ and η are primitive m-th and n-th roots of 1, respectively.

Hint. (i) Show that the mapping $\phi_j : g^k \to \varepsilon^{jk}$, $k = 0, 1, \dots, n-1$, is a one-dimensional representation of G. By Theorem 12.1.15, the group G has exactly n irreducible representations with pairwise different characters. Use similar arguments for (ii).

Exercise 12.1.18 Let D be the *dihedral group* of order 8, i.e. D is the *group of symmetries of the square* (all congruencies of the plane which fix a given square); D is generated by a rotation r and a reflection s with defining relations

$$r^4 = s^2 = 1,\ srs = r^{-1}.$$

(i) Show that $D = \{1, r, r^2, r^3, s, rs, r^2s, r^3s\}$ and the conjugacy classes of D are

$$\{1\},\ \{r^2\},\ \{r, r^3\},\ \{s, r^2s\},\ \{rs, r^3s\}.$$

(ii) Show that the table of irreducible characters of D coincides with the table given in Table 12.1.

Table 12.1. The character table of the dihedral group of order 8

D	1	r^2	r	s	rs
χ_1	1	1	1	1	1
χ_2	1	1	−1	1	1
χ_3	1	1	1	−1	−1
χ_4	1	1	−1	−1	1
χ_5	2	−2	0	0	0

Hint. (ii) Every representation of the factor group

$$\bar{D} = D/\langle r^2 \rangle \cong \langle \bar{r} \mid \bar{r}^2 = \bar{1} \rangle \times \langle \bar{s} \mid \bar{s}^2 = \bar{1} \rangle$$

can be considered as a representation of D. In this way, Exercise 12.1.17 gives the first four irreducible characters. The group D has an irreducible two-dimensional representation ϕ as the group of symmetries of the square. Fixing a basis of the two-dimensional vector space, we may assume that

$$\phi(r) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \ \phi(s) = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

and concrete calculations give the fifth irreducible character of D.

Exercise 12.1.19 Let H be the *quaternion group*, i.e. $H = \{\pm 1, \pm i, \pm j, \pm k\}$ and the multiplication of H is given by

$$i^2 = j^2 = k^2 = -1, \ ij = -ji = k, \ jk = -kj = i, \ ki = -ik = j.$$

(i) Show that the conjugacy classes of H are

$$\{1\}, \{-1\}, \{\pm i\}, \{\pm j\}, \{\pm k\}.$$

(ii) Show that the table of irreducible characters of H is the given in Table 12.2.

Table 12.2. The character table of the quaternion group

H	1	-1	i	j	k
χ_1	1	1	1	1	1
χ_2	1	1	−1	1	1
χ_3	1	1	1	−1	−1
χ_4	1	1	−1	−1	1
χ_5	2	−2	0	0	0

Hint. (ii) The factor group

$$\bar{H} = H/\langle -1 \rangle \cong \langle \bar{i} \mid \bar{i}^2 = \bar{1} \rangle \times \langle \bar{j} \mid \bar{j}^2 = \bar{1} \rangle$$

gives the first four irreducible characters. The group H has an irreducible two-dimensional representation ϕ defined (for $K = \mathbb{C}$) by

$$\phi(i) = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix},\ \phi(j) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

(where $i^2 = -1$ in $\mathbb{C}$) which gives the fifth irreducible character of H.

Comparing the character tables of the dihedral group of order 8 and the quaternion group given in Exercises 12.1.18 and 12.1.19, we see that D and H have the same tables of irreducible characters, although D and H are not isomorphic. Compare this with Theorem 12.1.15, where for a fixed finite group the character determines the representation.

Exercise 12.1.20 Let the field K be algebraically closed and let ρ be the regular representation of the finite group G.

(i) Show that $\chi_\rho(1) = |G|$ and $\chi_\rho(g) = 0$, if $g \neq 1$.

(ii) If $\chi_1, \ldots, \chi_r$ are all irreducible characters of G and d_i is the degree of χ_i, prove that

$$\chi_\rho = d_1\chi_1 + \ldots + d_r\chi_r.$$

Hint. (i) Calculate the trace of the matrix $\rho(g)$ with respect to the basis $\{g \mid g \in G\}$ of KG. (ii) Apply Maschke Theorem 12.1.6 and Exercise 12.1.8. Use that the matrix algebra $M_d(K)$ is a direct sum of d minimal left ideals, which are d-dimensional.

12.2 The Symmetric Group

In this section we summarize the necessary background on the representation theory of the symmetric group S_n.

Exercise 12.2.1 Let σ be a permutation of the symmetric group S_n presented as a product of independent cycles

$$\sigma = (i_1 \ldots i_p)(j_1 \ldots j_q) \ldots (k_1 \ldots k_r).$$

Show that the conjugacy class σ^{S_n} consists of all permutations with the same cyclic decomposition as σ.

Hint. Show that $\sigma = \tau\rho\tau^{-1}$, where

$$\rho = (1 \ldots p)(p+1 \ldots p+q) \ldots (n-r+1 \ldots n),$$

$$\tau = \begin{pmatrix} 1 & \ldots & p & p+1 & \ldots & p+q & \ldots & n-r+1 & \ldots & n \\ i_1 & \ldots & i_p & j_1 & \ldots & j_q & \ldots & k_1 & \ldots & k_r \end{pmatrix}.$$

Definition 12.2.2 A *partition* of n (in not more than k parts) is a k-tuple of nonnegative integers $\lambda = (\lambda_1, \ldots, \lambda_k)$ in decreasing order (i.e. $\lambda_1 \geq \ldots \geq$

$\lambda_k \geq 0$) and such that $\lambda_1 + \ldots + \lambda_k = n$. We use the notation $\lambda \vdash n$ or $\lambda \in \mathrm{Part}(n)$. We identify two partitions if they differ by a string of zeros in the end. For example, $\lambda = (4,2,1) = (4,2,1,0) = (4,2,1,0,0) \vdash 7$. It is also convenient to indicate the number of times each integer occurs in the partition and to write for example $(3, 2^2, 1^4)$ instead of $(3,2,2,1,1,1,1)$.

Definition 12.2.3 The *Young diagram* $[\lambda]$ of a partition $\lambda = (\lambda_1, \ldots, \lambda_k) \vdash n$ can be formally defined as the set of knots or points $(i,j) \in \mathbb{Z}^2$ such that $1 \leq j \leq \lambda_i$, $i = 1, \ldots, k$. Graphically we draw the diagrams replacing the knots by square boxes, adopting the convention, as with matrices, that the first coordinate i (the row index) increases as one goes downwards, and the second coordinate j (the column index) increases as one goes from left to right. The first boxes from the left of each row are one above another and the i-th row contains λ_i boxes. We denote by λ'_j the length of its j-th column of $[\lambda]$. The partition $\lambda' = (\lambda'_1, \ldots, \lambda'_l)$ and its diagram $[\lambda']$ are called *conjugate* respectively to λ and $[\lambda]$. For example, for $\lambda = (4,2,1)$, the corresponding Young diagram is given in Fig. 12.1 and $(4,2,1)' = (3,2,1^2)$.

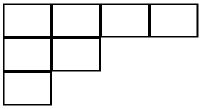


Fig. 12.1. The Young diagram $[\lambda] = [4,2,1]$

Definition 12.2.4 The (i,j)-*hook* of the diagram $[\lambda] = [\lambda_1, \ldots, \lambda_k]$ consists of the j-th box of the i-th row of $[\lambda]$ along with the $\lambda_i - j$ boxes to the right of it (called the *arm* of the hook) and the $\lambda'_j - i$ boxes below of it (the *leg* of the hook). The *length* of the hook is equal to $\lambda_i + \lambda'_j - i - j + 1$. For example, the $(2,3)$-hook of $[4^3, 3, 1]$ is of length 4, see Fig. 12.2.

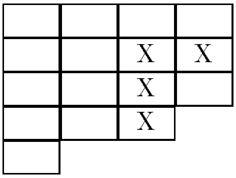


Fig. 12.2. The boxes of the $(2,3)$-hook of $[4^3, 3, 1]$ are marked by X

Definition 12.2.5 For a partition $\lambda = (\lambda_1, \ldots, \lambda_k) \vdash n$, we define a λ-*tableau* T_λ *of content* $\alpha = (\alpha_1, \ldots, \alpha_m)$, where $\alpha_1 + \ldots + \alpha_m = n$, as the

Young diagram $[\lambda]$ whose boxes are filled in with α_1 numbers 1, α_2 numbers 2, $\dots, \alpha_m$ numbers m. The tableau is *semistandard* if its entries do not decrease from left to right in the rows and increase from top to bottom in the columns. The tableau T_λ is *standard* if it is semistandard and of content $(1,\dots,1)$, i.e. every integer $1,\dots,n$ occurs in it exactly ones. For example, in Fig. 12.3, the two tableaux from the left are, respectively, semistandard of content $(2,3,1,1)$ and standard and the third is not semistandard.

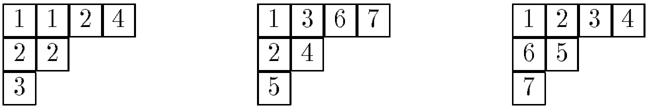


Fig. 12.3. The left and middle tableaux are semistandard, and the right is not

Definition 12.2.6 For a partition λ of n, the symmetric group S_n acts on the set of λ-tableaux of content $(1,\dots,1)$ in the following way. If the (i,j)-th box of the tableau T_λ contains the integer k, then the (i,j)-th box of σT_λ contains $\sigma(k)$, $\sigma \in S_n$. The *row stabilizer* $R(T_\lambda)$ of T_λ is the subgroup of S_n which fixes the sets of entries of each row of T_λ. Similarly one defines the *column stabilizer* $C(T_\lambda)$ of the tableau. For example, in Fig. 12.4 we give the $(3,2)$-tableau T and its image under the action of $\sigma = (12)(345)$. The row stabilizer of T is the direct product of S_3 acting on $1,4,2$ and S_2 acting on $5,3$. The column stabilizer is $C(T) = \langle (15) \rangle \times \langle (43) \rangle$.

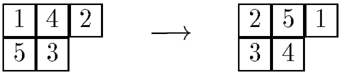


Fig. 12.4. The right tableau is the image of the left under $\sigma = (12)(345)$

The following theorem describes the irreducible representations of the symmetric group.

Theorem 12.2.7 *Let K be any field of characteristic 0. For a partition λ of n and a λ-tableau T_λ of content $(1,\dots,1)$, let $R(T_\lambda)$ and $C(T_\lambda)$ be, respectively, the row and column stabilizers of T_λ. Consider the element of the group algebra KS_n*

$$e(T_\lambda) = \sum_{\rho \in R(T_\lambda)} \sum_{\gamma \in C(T_\lambda)} (\mathrm{sign}\gamma)\rho\gamma.$$

(i) *Up to a multiplicative constant* $e(T_\lambda)$ *is equal to a minimal idempotent of the group algebra* KS_n *and generates a minimal left ideal of* KS_n *(i.e. an irreducible* S_n*-module).*

(ii) *If* μ *is another partition of* n *and* T_μ *is a* μ*-tableau, then the* S_n*-modules* $KS_n \cdot e(T_\lambda)$ *and* $KS_n \cdot e(T_\mu)$ *are isomorphic if and only if* $\lambda = \mu$.

(iii) *Every irreducible* S_n*-module is isomorphic to* $KS_n \cdot e(T_\lambda)$ *for some partition* $\lambda \vdash n$.

We denote respectively by $M(\lambda)$ *and* χ_λ *the irreducible* S_n*-module related with the partition* $\lambda \vdash n$ *and its character.*

Remark 12.2.8 For $\lambda \vdash n$, let T_1 and T_2 be two λ-tableaux. An isomorphism between the S_n-modules $KS_n \cdot e(T_1)$ and $KS_n \cdot e(T_2)$ is defined by $x \to x\sigma^{-1}$, $x \in KS_n \cdot e(T_1)$, where $\sigma \in S_n$ is such that $T_2 = \sigma T_1$. By the Schur lemma (Exercise 12.1.10 (ii)), all isomorphisms are $\phi_\alpha : KS_n \cdot e(T_1) \to KS_n \cdot e(T_2)$, where $\phi_\alpha(x) = \alpha x \sigma^{-1}$ and $0 \neq \alpha \in K$.

Remark 12.2.9 By Exercise 12.2.1, the conjugacy classes of S_n are determined by the cyclic decomposition of the permutations. We can relate to the permutation

$$\rho = (1 \ldots p)(p+1 \ldots p+q) \ldots (n-r+1 \ldots n), p \geq q \geq \ldots \geq r,$$

the partition $\lambda = (p, q, \ldots, r) \vdash n$. In this way we define a 1-1 correspondence between the conjugacy classes of S_n and the partitions of n, which agrees with Theorem 12.1.15 (ii). One of the differences between representation theory of the symmetric group and of some other groups is that the number of nonequivalent irreducible representations of S_n does not depend on the base field K, i.e. we do not need the assumption that K is sufficiently large (e.g. algebraically closed).

Exercise 12.2.10 Let $n = 3$. Find bases (as vector spaces) of the S_3-modules $KS_3 \cdot e(T_1)$ and $KS_3 \cdot e(T_2)$, where the $(2,1)$-tableaux T_1 and T_2 are respectively the left and the right tableaux in Fig. 12.5. Find an S_3-module isomorphism between $KS_3 \cdot e(T_1)$ and $KS_3 \cdot e(T_2)$. Find a representative of each S_3-submodule of KS_3 isomorphic to $M(2,1)$.

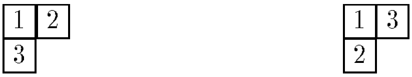


Fig. 12.5. The $(2,1)$-tableaux T_1 and T_2 in Exercise 12.2.10

Solution. The row and column stabilizers of T_1 are, respectively, $R(T_1) = \langle (12) \rangle$ and $C(T_1) = \langle (13) \rangle$. Hence

$$v_1 = e(T_1) = (1 + (12))(1 - (13)) = 1 + (12) - (13) - (132).$$

Acting on v_1 by the elements of S_3 we see that every element $\pi \cdot e(T_1)$, $\pi \in S_3$, is a linear combination of v_1 and

$$v_2 = (123) \cdot e(T_1) = (123) + (13) - (23) - 1.$$

Hence $M(2,1)$ is two-dimensional. Similarly we obtain a basis of $KS_3 \cdot e(T_2)$

$$w_1 = e(T_2) = (1 + (13))(1 - (12)) = 1 + (13) - (12) - (123),$$

$$w_2 = (123) \cdot e(T_2) = (123) + (23) - (13) - (132).$$

Since $T_2 = (23)T_1$, Remark 12.2.8 shows that an isomorphism between $KS_3 \cdot e(T_1)$ and $KS_3 \cdot e(T_2)$ is given by

$$v_1 \to v_1(23)^{-1} = v_1(23) = (23) + (123) - (132) - (13) = w_2,$$

$$v_2 \to v_2(23) = (12) + (132) - 1 - (23) = -w_1 - w_2.$$

Hence all isomorphisms $\phi_\alpha : KS_3 \cdot e(T_1) \longrightarrow KS_3 \cdot e(T_2)$ are defined by

$$\phi_\alpha(v_1) = \alpha w_2,\ \phi_\alpha(v_2) = \alpha(-w_1 - w_2),\ 0 \neq \alpha \in K.$$

Let $\phi : M' \longrightarrow M''$ be an isomorphism between the irreducible S_3-modules M' and M''. Using the Schur lemma (Exercise 12.1.10 (ii)), it is easy to see that every irreducible S_3-submodule of $M' \oplus M''$ is

$$M_{\xi\eta} = \{(\xi x, \eta\phi(x)) \mid x \in M'\},\ (\xi, \eta) \neq (0, 0),$$

and $M_{\xi\eta} = M_{\xi'\eta'}$ if and only if (ξ, η) and (ξ', η') are proportional. Since $\dim M(2,1) = 2$, Exercise 12.1.20 gives that KS_3 contains a direct sum of exactly two isomorphic copies of $M(2,1)$. Hence, the elements $v_1 + \alpha w_2$, $\alpha \in K$, and w_2 can be chosen as representatives of the irreducible submodules $M(2,1)$ of KS_3.

Exercise 12.2.11 Fill in the table of the irreducible characters of S_n for $n \leq 4$.

Hint. For $n = 1$ and $n = 2$ apply Exercise 12.1.17. For $n = 3$ use ideas similar to those of Exercise 12.1.18, bearing in mind that the symmetric group S_3 is isomorphic to the dihedral group of order 6 (the group of symmetries of the regular triangle) or working in the following way. For any $n > 1$ the group S_n has two one-dimensional representations, the trivial $\phi_1(\sigma) = 1$ and the *sign representation* $\phi_2(\sigma) = \mathrm{sign}\sigma$, $\sigma \in S_n$. Using Theorem 12.2.7, one can see that the trivial representation corresponds to the partition (n) and the sign representation to the partition (1^n). A third representation is obtained from the action of the symmetric group on the vector space V_n with basis $\{x_1, \ldots, x_n\}$ by $\sigma(x_i) = x_{\sigma(i)}$, $\sigma \in S_n$. It is easy to see that V_n is a direct sum of two irreducible S_n-modules, one of them the trivial, spanned on

$x_1+\ldots+x_n$, and the other $n-1$-dimensional. Calculate the character of V_n modulo the trivial representation. For $n=4$ use that the factor group of S_4 modulo the Kleinian subgroup $\{(ij)(kl)\in S_4\}$ is isomorphic to S_3 and the three irreducible representations of S_3 are also irreducible representations of S_4. The three-dimensional submodule of V_4 gives the fourth irreducible representation. Finally, we complete the character table by the tensor product of the sign representation and the three-dimensional representation which we just found. The character tables of S_3 and S_4 labeled, respectively, by the partitions of 3 and 4, are given in Tables 12.3 and 12.4.

Table 12.3. The character table of S_3

S_3	1	(12)	(123)
(3)	1	1	1
(2, 1)	2	0	−1
(1^3)	1	−1	1

Table 12.4. The character table of S_4

S_4	1	(12)	(123)	(1234)	(12)(34)
(4)	1	1	1	1	1
(3, 1)	3	1	0	−1	−1
(2^2)	2	0	−1	0	2
($2, 1^2$)	3	−1	0	1	−1
(1^4)	1	−1	1	−1	1

The degrees of the irreducible representations of S_n can be obtained in two ways.

Theorem 12.2.12 *Let λ be a partition of n.*

(i) *The dimension $d_\lambda=\dim M(\lambda)$ of the irreducible S_n-module $M(\lambda)$ is equal to the number of standard λ-tableaux.*

(ii) (The Hook Formula)

$$d_\lambda = \frac{n!}{\prod(\lambda_i + \lambda'_j - i - j + 1)},$$

where the product runs on all boxes $(i, j) \in [\lambda]$*, i.e. the denominator is equal to the product of the lengths of all hooks of the diagram* $[\lambda]$*.*

Exercise 12.2.13 Calculate the degrees of the irreducible representations of S_5.

Hint. (i) Applying Theorem 12.2.12 (i), we obtain, for example, that there are five standard $(3, 2)$-tableaux given in Fig. 12.6 and $\dim M(3, 2) = 5$.

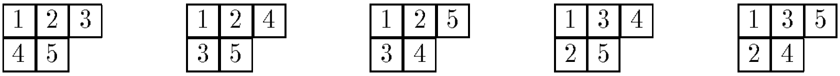


Fig. 12.6. The standard $(3, 2)$-tableaux

(ii) Applying the hook formula of Theorem 12.2.12 (ii), we obtain the lengths of the hooks of the diagram $[3, 2]$ (written in the boxes of the diagram in Fig. 12.7).

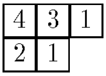


Fig. 12.7. The lengths of the hooks of the diagram $[3, 2]$

Hence

$$\dim M(3, 2) = \frac{1.2.3.4.5}{4.3.1.2.1} = 5.$$

Similarly we obtain for the other irreducible S_5-modules

$$\dim M(5) = 1,\ \dim M(4, 1) = 4,\ \dim M(3, 1^2) = 6,$$

$$\dim M(2^2, 1) = 5,\ \dim M(2, 1^3) = 4,\ \dim M(1^5) = 1.$$

Remark 12.2.14 In Table 12.5 we give the character table of the symmetric group S_5 which we need in some of the exercises below.

Table 12.5. The character table of S_5

S_5	1	(12)	(123)	(1234)	(12)(34)	(123)(45)	(12345)
(5)	1	1	1	1	1	1	1
(4,1)	4	2	1	0	0	−1	−1
(3,2)	5	1	−1	−1	1	1	0
$(3,1^2)$	6	0	0	0	−2	0	1
$(2^2,1)$	5	−1	−1	1	1	−1	0
$(2,1^3)$	4	−2	1	0	0	1	−1
(1^5)	1	−1	1	−1	1	−1	1

12.3 Multilinear Polynomial Identities

This section is devoted to some applications of representation theory of the symmetric groups to PI-algebras. The main idea is to use the action of the symmetric group S_n on the vector space of multilinear polynomials of degree n and to translate and solve the considered problems on PI-algebras in the language of representation theory of S_n. This action of S_n is given in the next exercise.

Exercise 12.3.1 Let P_n be the set of all multilinear polynomials of degree n in the free associative algebra $K\langle X\rangle$.

(i) Show that the following action of the symmetric group S_n makes P_n a left S_n-module isomorphic to the group algebra KS_n considered as a left S_n-module (related to the regular representation of S_n):

$$\sigma(\sum \alpha_i x_{i_1}\dots x_{i_n}) = \sum \alpha_i x_{\sigma(i_1)}\dots x_{\sigma(i_n)},\ \sigma\in S_n,\ \alpha_i\in K,\ x_{i_1}\dots x_{i_n}\in P_n.$$

(ii) If U is a T-ideal of $K\langle X\rangle$, prove that $U\cap P_n$ is a submodule of P_n.

Hint. (i) The vector space P_n has a basis

$$\{x_{\pi(1)}\dots x_{\pi(n)}\mid \pi\in S_n\}$$

and, identifying π with $x_{\pi(1)}\dots x_{\pi(n)}$ we obtain an isomorphism of KS_n and P_n as vector spaces. Show that this is also an S_n-module isomorphism. (ii) Use that the T-ideal U is invariant under all substitutions and for $f(x_1,\ldots,x_n)\in U\cap P_n$

$$\sigma(f(x_1,\ldots,x_n))=f(x_{\sigma(1)},\ldots,x_{\sigma(n)})\in U\cap P_n,\ \sigma\in S_n.$$

In order to apply successfully representation theory of S_n to concrete problems we have to know the module structure of some important submodules of the S_n-module P_n. The next exercises contain some typical calculations. (In the next sections we shall see how some of the considerations can be simplified using representation theory of the general linear group.)

Exercise 12.3.2 Let $L(X)$ be the free Lie algebra considered as a Lie subalgebra of the free associative algebra $K\langle X\rangle$ and let $PL_n=P_n\cap L(X)$ be the set of multilinear Lie polynomials of degree n. Determine the S_n-module structure of PL_n for $n\leq 3$.

Solution. For $n=1$ the vector space PL_1 is one-dimensional and S_1 has the trivial irreducible representation only. Hence $PL_1\cong M(1)$.

For $n=2$, again $\dim PL_2=1$ because PL_2 is spanned on $[x_1,x_2]$. Since

$$(12)[x_1,x_2]=-[x_1,x_2],$$

we obtain that this is the sign representation of S_2. Hence $PL_2\cong M(1^2)$.

Now, let $n=3$. We shall consider several possibilities to handle the problem.

Method 1. We shall show directly, that PL_3 is an irreducible S_3-module. Let M be an irreducible submodule of PL_3 and let $0\neq f\in M$. Every element of PL_3 has the form

$$f=f(x_1,x_2,x_3)=\alpha[x_2,x_1,x_3]+\beta[x_3,x_1,x_2],\ \alpha,\beta\in K.$$

Since $[x_3,x_2,x_1]=-[x_2,x_1,x_3]+[x_3,x_1,x_2]$,

$$(12)f=-(\alpha+\beta)[x_2,x_1,x_3]+\beta[x_3,x_1,x_2]\in M$$

and if $\beta\neq 0$ and $2\alpha+\beta\neq 0$, we obtain that both $[x_2,x_1,x_3]$ and $[x_3,x_1,x_2]$ are in M. Hence $M=PL_3$. If $\beta=0$, then $f=\alpha[x_2,x_1,x_3]$ and $(23)f=\alpha[x_3,x_1,x_2]\in M$, i.e. again $M=PL_3$. Finally, if $2\alpha+\beta=0$, we may assume that $\alpha=1$, $\beta=-2$ and

$$f=[x_2,x_1,x_3]-2[x_3,x_1,x_2]\in M,\ (23)f=-2[x_2,x_1,x_3]+[x_3,x_1,x_2]\in M.$$

Again, f and $(23)f$ are linearly independent and $M=PL_3$. Having a look in the character table of S_3 (see Exercise 12.2.11 and Table 12.3) we make the observation that the only irreducible two-dimensional S_3-module is $M(2,1)$. Hence $PL_3\cong M(2,1)$.

Method 2. We shall calculate the character χ of PL_3. Since $\dim PL_3 = 2$, we obtain $\chi(1) = \dim PL_3 = 2$. Further,

$$(23)[x_2, x_1, x_3] = [x_3, x_1, x_2],\ (23)[x_3, x_1, x_2] = [x_2, x_1, x_3],\ \chi(23) = 0,$$

$$(123)[x_2, x_1, x_3] = [x_3, x_2, x_1] = -[x_2, x_1, x_3] + [x_3, x_1, x_2],$$

$$(123)[x_3, x_1, x_2] = -[x_2, x_1, x_3],\ \chi(123) = -1.$$

Hence $\chi = \chi_{(2,1)}$. Another possibility is to assume that

$$\chi = m(3)\chi_{(3)} + m(2,1)\chi_{(2,1)} + m(1^3)\chi_{(1^3)}$$

for some nonnegative integers $m(3)$, $m(2,1)$ and $m(1^3)$. Using the character table of S_3 we obtain the following linear system with unknowns $m(3)$, $m(2,1)$ and $m(1^3)$:

$$\chi(1) = 1 \cdot m(3) + 2 \cdot m(2,1) + 1 \cdot m(1^3) = 2$$

$$\chi(12) = 1 \cdot m(3) + 0 \cdot m(2,1) - 1 \cdot m(1^3) = 0$$

$$\chi(123) = 1 \cdot m(3) - 1 \cdot m(2,1) + 1 \cdot m(1^3) = -1.$$

The only solution of the system is

$$m(3) = 0,\ m(2,1) = 1,\ m(1^3) = 0,$$

and this gives again that $PL_3 \cong M(2,1)$.

Method 3. Let $\iota : KS_3 \longrightarrow P_3$ be the S_3-module isomorphism in Exercise 12.3.1. We make use of Exercise 12.2.10 and use its notation. For every S_3-submodule $M(2,1)$ of P_3 we may choose a representative which is equal either to $\iota(v_1 + \alpha w_2)$ for some $\alpha \in K$ or to $\iota(w_2)$, where

$$\iota(v_1) = \iota(1 + (12) - (13) - (132)) = x_1x_2x_3 + x_2x_1x_3 - x_3x_2x_1 - x_3x_1x_2,$$

$$\iota(w_2) = \iota((123) + (23) - (13) - (132)) = x_2x_3x_1 + x_1x_3x_2 - x_3x_2x_1 - x_3x_1x_2.$$

We are looking for $\alpha \in K$, such that $\iota(v_1 + \alpha w_2) \in PL_3$. This gives the condition

$$x_1x_2x_3 + x_2x_1x_3 - x_3x_2x_1 - x_3x_1x_2 + \alpha(x_2x_3x_1 + x_1x_3x_2 - x_3x_2x_1 - x_3x_1x_2) =$$

$$= \xi[x_2, x_1, x_3] + \eta[x_3, x_1, x_2],\ \xi, \eta \in K.$$

Since

$$[x_2, x_1, x_3] = x_2x_1x_3 - x_1x_2x_3 - x_3x_2x_1 + x_3x_1x_2,$$

$$[x_3, x_1, x_2] = x_3x_1x_2 - x_1x_3x_2 - x_2x_3x_1 + x_2x_1x_3,$$

comparing the coefficients of $x_ix_jx_k$, we obtain that $\alpha = -2$, $\xi = -1$, $\eta = 2$, i.e.

$$\iota(v_1) - 2\iota(w_2) = -[x_2, x_1, x_3] + 2[x_3, x_1, x_2]$$

and this means that PL_3 contains a submodule $M(2,1)$. Since

$$\mathrm{dim} PL_3 = \mathrm{dim} M(2,1) = 2,$$

we obtain that $PL_3 \cong M(2,1)$.

Exercise 12.3.3 Determine the S_4-module structure of the sets of the multilinear Lie elements PL_4 and of the proper multilinear elements Γ_4 in $K\langle X\rangle$.

Hint. (i) Use the basis of PL_4 consisting of the following elements

$$[x_2,x_1,x_3,x_4],\ [x_3,x_1,x_2,x_4],\ [x_4,x_1,x_2,x_3],$$

$$[[x_2,x_1],[x_4,x_3]],\ [[x_3,x_1],[x_4,x_2]],\ [[x_4,x_1],[x_3,x_2]],$$

and calculate the character χ of PL_4. Use the character table of S_4 (Exercise 12.2.11 and Table 12.4) to decompose χ as a sum of irreducible characters. The final result is

$$PL_4 \cong M(3,1) \oplus M(2,1^2).$$

(ii) Use the basis of Γ_4 consisting of

$$[x_i,x_1,x_2,\ldots,\hat{x}_i,\ldots,x_4],\ i=2,3,4,$$

$$[x_i,x_j][x_k,x_l],\ i>j,\ k>l,$$

(see Theorem 5.2.1). It is better to simplify the calculations in the following way. Consider the factor module Γ_4/PL_4. It has a basis

$$\{[x_2,x_1]\circ[x_4,x_3],\ [x_3,x_1]\circ[x_4,x_2],\ [x_4,x_1]\circ[x_3,x_2]\},$$

where $u\circ v = uv+vu$. Calculate its character and show that $\Gamma_4/PL_4 \cong M(2^2)\oplus M(1^4)$. Derive from here that

$$\Gamma_4 \cong M(3,1)\oplus M(2^2)\oplus M(2,1^2)\oplus M(1^4).$$

Exercise 12.3.4 Determine the S_5-module structure of the sets of the multilinear Lie elements PL_5 and of the proper multilinear elements Γ_5 in $K\langle X\rangle$.

Hint. Use the character table of S_5 in Remark 12.2.14 and Table 12.5. The answer is

$$PL_5 \cong M(4,1)\oplus M(3,2)\oplus M(3,1^2)\oplus M(2^2,1)\oplus M(2,1^3),$$

$$\Gamma_5 \cong M(4,1)\oplus 2M(3,2)\oplus 2M(3,1^2)\oplus 2M(2^2,1)\oplus 2M(2,1^3).$$

When we consider the applications of representation theory of the general linear group we shall give another method for decomposing the Lie and the proper multilinear polynomials of small degree.

Remark 12.3.5 There are several descriptions of the S_n-module structure of the set of multilinear Lie elements PL_n. In the paper of Klyachko [150] this description is given in the language of the so called *induced characters* and in the paper of Thrall [251] in the language of *plethysms*. See also the book by Bahturin [21].

Below we define the cocharacter sequence of a PI-algebra or of the corresponding T-ideal. This is one of the most important invariants of the PI-algebra which carries almost all necessary quantitative information about its polynomial identities.

Definition 12.3.6 Let R be a PI-algebra and let $P_n(R) = P_n/(T(R) \cap P_n)$, $n = 0, 1, 2, \dots$ The S_n-character

$$\chi_n(R) = \chi_{P_n(R)} = \sum_{\lambda \vdash n} m_\lambda(R)\chi_\lambda$$

is called the *n-cocharacter* of the polynomial identities of the algebra R. The sequence

$$\chi_n(R),\ n = 0, 1, 2, \dots,$$

is called the *cocharacter sequence* of R.

Since the n-th codimension $c_n(R)$ of the PI-algebra R is equal to the dimension of the S_n-module $P_n(R)$, we obtain immediately that $c_n(R)$ is equal to the evaluation of $\chi_n(R)$ on the identity permutation.

Exercise 12.3.7 Show that for any (unitary) commutative algebra R

$$\chi_n(R) = \chi_{(n)},\ n = 0, 1, 2, \dots,$$

i.e. the cocharacter sequence of R consists of trivial characters only.

Hint. Since the T-ideal of R coincides with the commutator ideal of $K\langle X\rangle$, the relatively free algebra $F(R)$ is isomorphic to the polynomial algebra $K[X]$ in infinitely many commuting variables. Hence $P_n(R)$ is spanned on the monomial $x_1 \dots x_n$ and

$$\sigma(x_1 \dots x_n) = x_1 \dots x_n,\ \sigma \in S_n,$$

i.e. $P_n(R)$ is the trivial module of S_n.

Now we shall give the Regev proof [223] of the Amitsur theorem [10] that every PI-algebra satisfies some power of a standard identity. The original proof of Amitsur is based on his theorem that the T-ideals $T(M_k(K))$ are the only prime T-ideals (see Remark 5.2.2) and the fact that the Jacobson radical of any PI-algebra is nil (which is a much easier result than the Razmyslov-Kemer-Braun theorem for the nilpotency of the radical of a finitely generated PI-algebra). The proof of the theorem of Amitsur can be found e.g. in the book of Rowen [231].

As in Theorem 12.2.7, for a partition λ of n and a standard λ-tableau T, we denote by $R(T)$ and $C(T)$, respectively, the row and column stabilizers of T, and assume that

$$e(T) = \sum_{\rho\in R(T)} \sum_{\gamma\in C(T)} (\mathrm{sign}\gamma)\rho\gamma.$$

We also denote by ι the S_n-module isomorphism in Exercise 12.3.1

$$\iota : KS_n \longrightarrow P_n.$$

Lemma 12.3.8 *Let $\lambda = (m^k)$ be a partition of $n = km$ in k equal parts and let the first column of the standard λ-tableau T be filled in with $1, 2, \dots, k$, the second column with $k+1, \dots, 2k$, ..., the m-th column with $(m-1)k + 1, \dots, km$. Let*

$$f(x_1, \dots, x_{km}) = \iota(e(T))$$

be the multilinear polynomial of P_{km} corresponding to $e(T)$. Then the homogeneous polynomial

$$w(x_1, \dots, x_k) = f(x_1, \dots, x_k, x_1, \dots, x_k, \dots, x_1, \dots, x_k)$$

obtained by identifying the variables with indices in the same row of T, i.e.

$$x_i = x_{k+i} = \dots = x_{(m-1)k+i},\ i = 1, \dots, k,$$

is equal to $(m!)^k s_k^m(x_1, \dots, x_k)$, where s_k is the standard polynomial of degree k.

Proof. The column stabilizer of T is

$$C(T) = S_k \times \dots \times S_k,$$

where the j-th copy of S_k acts on $\{(j-1)k + 1, \dots, jk\}$. Hence

$$\iota\left(\sum_{\gamma\in C(T)} (\mathrm{sign}\gamma)\gamma\right) =$$

$$= s_k(x_1, \dots, x_k)s_k(x_{k+1}, \dots, x_{2k}) \dots s_k(x_{(m-1)k+1}, \dots, x_{mk}).$$

Since ι is an S_{km}-module isomorphism,

$$\iota(e(T)) = \iota\left(\sum_{\rho\in R(C)} \rho \sum_{\gamma\in C(T)} (\mathrm{sign}\gamma)\gamma\right) = \sum_{\rho\in R(C)} \rho\iota\left(\sum_{\gamma\in C(T)} (\mathrm{sign}\gamma)\gamma\right).$$

The row stabilizer $R(T)$ is a direct product of k copies $S_m^{(i)}$ of symmetric groups, the group $S_m^{(i)}$ acting on $\{i, k+i, \dots, (m-1)k + i\}$. Hence, for fixed i, the variables $x_i, x_{k+i}, \dots, x_{(m-1)k+i}$ are symmetric in

$$\sum_{\rho\in S_m^{(i)}} \rho\iota\left(\sum_{\gamma\in C(T)}(\mathrm{sign}\gamma)\gamma\right),$$

and the substitution

$$x_i = x_{k+i} = \ldots = x_{(m-1)k+i},\ i = 1,\ldots,k,$$

gives

$$w(x_1,\ldots,x_k) = (m!)^k s_k^m(x_1,\ldots,x_k).$$

Example 12.3.9 If $k = m = 2$, then T is given in Fig. 12.8 and

$$\begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & 4 \\ \hline\end{array}$$

Fig. 12.8. The tableau T for $m = k = 2$

$$\iota\left(\sum_{\gamma\in C(T)}(\mathrm{sign}\gamma)\right) = (1-(12))(1-(34))\cdot x_1x_2x_3x_4 =$$
$$= (x_1x_2 - x_2x_1)(x_3x_4 - x_4x_3) = s_2(x_1,x_2)s_2(x_3,x_4),$$
$$f(x_1,x_2,x_3,x_4) = \iota(e(T)) = \sum_{\rho\in R(T)} \rho\cdot s_2(x_1,x_2)s_2(x_3,x_4) =$$
$$= (1+(13))(1+(24))\cdot s_2(x_1,x_2)s_2(x_3,x_4) =$$
$$= s_2(x_1,x_2)s_2(x_3,x_4) + s_2(x_3,x_2)s_2(x_1,x_4) +$$
$$+ s_2(x_1,x_4)s_2(x_3,x_2) + s_2(x_3,x_4)s_2(x_1,x_2),$$
$$f(x_1,x_2,x_1,x_2) = (2!)^2 s_2^2(x_1,x_2).$$

Theorem 12.3.10 (Amitsur [10]) *Every PI-algebra satisfies the polynomial identity*

$$s_k^m(x_1,\ldots,x_k) = \left(\sum_{\sigma\in S_k}(\mathrm{sign}\sigma)x_{\sigma(1)}\ldots x_{\sigma(k)}\right)^m = 0$$

for some $k, m \geq 1$.

Proof. We give the proof of Regev [223]. Let the PI-algebra R satisfy a polynomial identity of degree d. By Regev Codimension Theorem 8.1.7, the codimension sequence of R satisfies the inequality

$$c_n(R) \leq (d-1)^{2n},\ n = 0, 1, 2, \dots$$

The idea of the proof is the following. We shall find k and m such that the dimension d_λ of the irreducible S_{km}-module $M(\lambda)$ corresponding to the partition $\lambda = (m^k)$ is bigger than $c_{km}(R)$. This implies that $M(\lambda)$ is not a submodule of the S_{km}-module $P_{km}(R)$. By Maschke Theorem 12.1.6, the following S_{km}-module isomorphism holds:

$$P_{km} \cong P_{km}(R) \oplus (P_{km} \cap T(R)).$$

Since $M(\lambda)$ is irreducible and participates in the decomposition of P_{km}, we obtain that all submodules of P_{km} isomorphic to $M(\lambda)$ belong to $P_{km} \cap T(R)$. In particular, the element $\iota(e(T))$ also belongs to $P_{km} \cap T(R)$, where T is the λ-tableau considered in Lemma 12.3.8. This gives that $s_k^m \in T(R)$, i.e. $s_k^m = 0$ is a polynomial identity for R.

We fix $m \in \mathbb{N}$ such that $m \geq 2(d-1)^2$ and allow $k \geq m$. We shall apply the hook formula in Theorem 12.2.12 (ii) for the dimension of the S_{km}-module $M(m^k)$. Since each row of the diagram $[\lambda] = [m^k]$ has m boxes and each column has k boxes, we obtain that the length of the (i,j)-th hook of $[\lambda]$ is $m + k - i - j + 1$ and

$$d_\lambda = (km)! \frac{(m-1)!(m-2)! \dots 1!0!}{(k+m-1)!(k+m-2)! \dots (k+1)!k!} > \frac{(km)!}{((k+m)!)^m}.$$

We use the Stirling formula for $n!$

$$n! = \sqrt{2\pi n} n^n e^{-n} e^{\theta(n)},\ |\theta(n)| < \frac{1}{12n},$$

or, for sufficiently large n,

$$n! \approx \sqrt{2\pi n} n^n e^{-n}.$$

We consider k sufficiently large and make θ in the Stirling formula close to 0. Since m is fixed, we obtain

$$d_\lambda > \frac{(km)!}{((k+m)!)^m} \approx \frac{\sqrt{2\pi km}(km)^{km} e^{-km}}{\left(\sqrt{2\pi(k+m)}(k+m)^{k+m} e^{-(k+m)}\right)^m}.$$

We have assumed that $k \geq m$. Hence $\sqrt{2\pi km} > 1$, $k + m \leq 2k$ and

$$d_\lambda > \frac{k^{km} m^{km} e^{m^2}}{2^m \pi^{m/2} k^{m/2} k^{(k+m)m} 2^{(k+m)m}} =$$

$$= \frac{\alpha}{k^{m^2+m/2}} \left(\frac{m}{2}\right)^{km} > \frac{\alpha}{k^{2m^2}} \left(\frac{m}{2}\right)^{km},\ \alpha = \frac{e^{m^2}}{2^{m^2+m} \pi^{m/2}}.$$

Since $m > 2(d-1)^2$ and $c_{km} < (d-1)^{mk}$, we obtain

$$\lim_{k\to\infty}\frac{d_\lambda}{c_{km}(R)}=\infty$$

and, for k sufficiently large,

$$d_\lambda > c_{km}(R).$$

As we have already seen, this implies that $s_k^m = 0$ is a polynomial identity for R.

Remark 12.3.11 (i) The original proof of Theorem 12.3.10 given by Amitsur [10] does not provide estimates for the values of k and m. In the proof of Regev presented above, one can obtain some bounds for k and m. For better bounds see the paper by Regev [223].

(ii) Since the degrees of the irreducible S_{km}-representations corresponding to the conjugate partitions m^k and k^m are equal, one can obtain that if $s_k^m = 0$ is a polynomial identity for the algebra R, where k and m are obtained from the proof of Regev, then $s_m^k = 0$ is also a polynomial identity for R.

Now we state without proof the theorem of Amitsur and Regev [12] about the partitions and the shapes of the Young diagrams corresponding to the irreducible characters in the cocharacter sequence of any PI-algebra. The proof is based on estimates similar to those in Theorem 12.3.10 and work with the elements $e(T)$ generating irreducible S_n-modules. I think that the reader is prepared to follow the proof in the original paper of Amitsur and Regev.

Theorem 12.3.12 (Amitsur-Regev [12]) *For every PI-algebra R there exist nonnegative integers k and l such that in the cocharacter sequence of R*

$$\chi_n(R)=\sum_{\lambda\vdash n} m_\lambda(R)\chi_\lambda,\ n=0,1,2,\ldots,$$

the partitions $\lambda=(\lambda_1,\ldots,\lambda_m)$ corresponding to nonzero multiplicities $m_\lambda(R)$ satisfy the condition $\lambda_{k+1}\leq l$. In other words, their diagrams $[\lambda]$ are in a hook with height k of the arm and wide l of the leg, see Fig. 12.9.

If we know the size of the hook of Theorem 12.3.12, we can estimate better the asymptotic behaviour of the codimension sequences of the PI-algebra. For example, one can show that if in the notation of Theorem 12.3.12 $\lambda_{k+1}\leq l$ for all λ with $m_\lambda(R)\neq 0$, then

$$\limsup_{n\to\infty}\sqrt[n]{c_n(R)}\leq k+l.$$

This result follows from the theorem of Berele and Regev [32] that the multiplicities $m_\lambda(R)$ are bounded by a polynomial of $n=|\lambda|$ and estimates similar to these in the proof of Theorem 12.3.10. Hooks which provide more detailed

Fig. 12.9 The diagrams $[\lambda]$ with $m_\lambda(R) \neq 0$ are in a hook

information about the nonzero multiplicities in the cocharacter sequence of the PI-algebra are studied by Popov [210].

12.4 The Action of the General Linear Group

In this section we survey the information on representation theory of the general linear group in a form which we need for our study of PI-algebras. We restrict most of our considerations to the case when $GL_m(K)$ acts on the free associative algebra of rank m. The main application of representation theory of $GL_m(K)$ in this section is the theorem of Berele and Drensky. Freely restated, it gives that any result on multilinear polynomial identities obtained in the language of representations of the symmetric group is equivalent to a corresponding result on homogeneous polynomial identities obtained in the language of representations of the general linear group.

Definition 12.4.1 Let ϕ be a finite dimensional representation of the general linear group $GL_m(K)$, i.e. $\phi : GL_m(K) \longrightarrow GL_s(K)$ for some s. The representation ϕ is *polynomial* if the entries $(\phi(g))_{pq}$ of the $s \times s$ matrix $\phi(g)$ are polynomials of the entries a_{kl} of g for $g \in GL_m(K)$, $k, l = 1, \ldots, m$, $p, q = 1, \ldots, s$. The polynomial representation ϕ is *homogeneous of degree* d if the polynomials $(\phi(g))_{pq}$ are homogeneous of degree d. The $GL_m(K)$-module W is called *polynomial* if the corresponding representation is polynomial. Similarly one introduces *homogeneous* polynomial modules.

As in Chapter 10, we fix the vector space V_m with basis $\{x_1, \ldots, x_m\}$ and with the canonical action of $GL_m(K)$. We also assume that

$$K\langle V_m\rangle = K\langle x_1,\ldots,x_m\rangle$$

is the free associative algebra of rank m.

The following exercise introduces the action of $GL_m(K)$ which we shall use till the end of the chapter (compare it with Exercise 12.3.1).

Exercise 12.4.2 Extend the action on V_m of $GL_m(K)$ diagonally on the free associative algebra $K\langle V_m\rangle$ by

$$g(x_{i_1}\ldots x_{i_n}) = g(x_{i_1})\ldots g(x_{i_n}),\ g\in GL_m(K),\ x_{i_1}\ldots x_{i_n}\in K\langle V_m\rangle.$$

(i) Show that $K\langle V_m\rangle$ is a left $GL_m(K)$-module which is a direct sum of its submodules $(K\langle V_m\rangle)^{(n)}$, $n=0,1,2,\ldots$, where $(K\langle V_m\rangle)^{(n)}$ is the homogeneous component of degree n of $K\langle V_m\rangle$.

(ii) Show that for every T-ideal U of $K\langle X\rangle$, the vector spaces $U\cap K\langle V_m\rangle$ and $U\cap (K\langle V_m\rangle)^{(n)}$ are submodules of $K\langle V_m\rangle$.

(iii) Show that every submodule W of $K\langle V_m\rangle$ is a direct sum of its homogeneous components $W\cap (K\langle V_m\rangle)^{(n)}$.

Hint. (i) Show that the action of $GL_m(K)$ is a module action and $(K\langle V_m\rangle)^{(n)}$ is $GL_m(K)$-invariant.

(ii) Use that if $f(x_1,\ldots,x_m)$ belongs to the T-ideal U and $g\in GL_m(K)$, then

$$g(f(x_1,\ldots,x_m)) = f(g(x_1),\ldots,g(x_m))\in U,$$

and $U\cap K\langle V_m\rangle$ is $GL_m(K)$-invariant. For $U\cap (K\langle V_m\rangle)^{(n)}$ use that the T-ideals of $K\langle X\rangle$ are homogeneous ideals.

(iii) If f_n is the homogeneous component of degree n of the polynomial f in W, show that the action of the scalar matrix αe, $0\neq\alpha\in K$, e being the identity matrix in $GL_m(K)$, multiplies f_n by α^n. Apply Vandermonde arguments to show the statement for the homogeneous components.

The polynomial representations of $GL_m(K)$ have many properties similar to those of the representations of finite groups.

Theorem 12.4.3 (i) *Every polynomial representation of $GL_m(K)$ is a direct sum of irreducible homogeneous polynomial subrepresentations.*

(ii) *Every irreducible homogeneous polynomial $GL_m(K)$-module of degree $n\geq 0$ is isomorphic to a submodule of $(K\langle V_m\rangle)^{(n)}$.*

The irreducible homogeneous polynomial representations of degree n of $GL_m(K)$ are described by partitions of n in not more than m parts and Young diagrams.

Theorem 12.4.4 (i) *The pairwise nonisomorphic irreducible homogeneous polynomial $GL_m(K)$-representations of degree $n\geq 0$ are in 1-1 correspondence with the partitions $\lambda=(\lambda_1,\ldots,\lambda_m)$ of n. We denote by $W_m(\lambda)$ the irreducible $GL_m(K)$-module related to λ.*

(ii) *Let* $\lambda = (\lambda_1, \dots, \lambda_m)$ *be a partition of* n*. The* $GL_m(K)$*-module* $W_m(\lambda)$ *is isomorphic to a submodule of* $(K\langle V_m\rangle)^{(n)}$*. The* $GL_m(K)$*-module* $(K\langle V_m\rangle)^{(n)}$ *has a decomposition*

$$(K\langle V_m\rangle)^{(n)} \cong \sum d_\lambda W_m(\lambda),$$

where d_λ *is the dimension of the irreducible* S_n*-module* $M(\lambda)$ *and the summation runs on all partitions* λ *of* n *in not more than* m *parts.*

(iii) *As a subspace of* $(K\langle V_m\rangle)^{(n)}$*, the vector space* $W_m(\lambda)$ *is multihomogeneous. The dimension of its multihomogeneous component* $W_m^{(n_1,\dots,n_m)}$ *is equal to the number of semistandard* λ*-tableaux of content* $(n_1, \dots, n_m)$*.*

(iv) *The Hilbert series*

$$\mathrm{Hilb}(W_m(\lambda), t_1, \dots, t_m) = \sum \mathrm{dim} W_m^{(n_1,\dots,n_m)} t_1^{n_1} \dots t_m^{n_m}$$

is a symmetric polynomial and is equal to the quotient

$$\frac{D(\lambda_1 + m - 1, \lambda_2 + m - 2, \dots, \lambda_{m-1} + 1, \lambda_m)}{D(m-1, m-2, \dots, 1, 0)},$$

where

$$D(\mu_1, \dots, \mu_m) = \begin{vmatrix} t_1^{\mu_1} & t_2^{\mu_1} & \dots & t_{m-1}^{\mu_1} & t_m^{\mu_1} \\ t_1^{\mu_2} & t_2^{\mu_2} & \dots & t_{m-1}^{\mu_2} & t_m^{\mu_2} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ t_1^{\mu_{m-1}} & t_2^{\mu_{m-1}} & \dots & t_{m-1}^{\mu_{m-1}} & t_m^{\mu_{m-1}} \\ t_1^{\mu_m} & t_2^{\mu_m} & \dots & t_{m-1}^{\mu_m} & t_m^{\mu_m} \end{vmatrix}.$$

The function

$$S_\lambda(t_1, \dots, t_m) = \mathrm{Hilb}(W_m(\lambda), t_1, \dots, t_m)$$

is called the Schur function corresponding to λ*.*

Example 12.4.5 Let $m = n = 3$ and let $\lambda = (2, 1)$. The semistandard $(2, 1)$-tableaux of content (n_1, n_2, n_3) for $n_1 + n_2 + n_3 = 3$, $n_1 \geq n_2 \geq n_3$, are given in Fig. 12.10. Hence there are two semistandard tableaux of content $(1, 1, 1)$ and one tableau of content $(2, 1, 0)$. Since $S_{(2,1)}(t_1, t_2, t_3)$ is a symmetric polynomial, we obtain that

$$S_{(2,1)}(t_1, t_2, t_3) = 2t_1t_2t_3 + t_1^2t_2 + t_1t_2^2 +$$
$$t_1^2t_3 + t_1t_3^2 + t_2^2t_3 + t_2t_3^2.$$

Using the determinant formula, we obtain

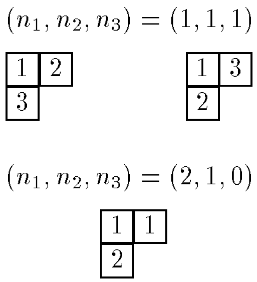


Fig. 12.10. The semistandard $(2,1)$-tableaux of content $(1,1,1)$ and $(2,1,0)$

$$D(2+2,1+1,0+0) = \begin{vmatrix} t_1^4 & t_2^4 & t_3^4 \\ t_1^2 & t_2^2 & t_3^2 \\ 1 & 1 & 1 \end{vmatrix}, \ D(2,1,0) = \begin{vmatrix} t_1^2 & t_2^2 & t_3^2 \\ t_1 & t_2 & t_3 \\ 1 & 1 & 1 \end{vmatrix},$$

$$S_{(2,1)}(t_1,t_2,t_3) = \frac{D(2+2,1+1,0+0)}{D(2,1,0)} =$$

$$= \frac{(t_1^2-t_2^2)(t_1^2-t_3^2)(t_2^2-t_3^2)}{(t_1-t_2)(t_1-t_3)(t_2-t_3)} = (t_1+t_2)(t_1+t_3)(t_2+t_3)$$

which gives the same answer.

Exercise 12.4.6 Show that

$$S_{(n)}(t_1,\ldots,t_m) = h_n(t_1,\ldots,t_m) = \sum t_1^{n_1}\ldots t_m^{n_m},\ n_1+\ldots+n_m = n,$$

is the *complete symmetric polynomial* of degree n,

$$S_{(1^n)}(t_1,\ldots,t_m) = e_n(t_1,\ldots,t_m) = \sum t_{i_1}\ldots t_{i_n},\ 1 \le i_1 < \ldots < i_n \le m,$$

is the elementary symmetric polynomial of degree $n \le m$.

Hint. The semistandard (n)-tableaux are in 1-1 correspondence with the monomials of degree n in $t_1,\ldots,t_m$. For $\lambda = (1^n)$ the correspondence is with the monomials of degree ≤ 1 in each variable.

Theorem 12.4.7 *Let* $W = \sum m_\lambda W_m(\lambda)$ *be a polynomial* $GL_m(K)$*-module,* $W_m(\lambda) \subset K\langle V_m\rangle$*. Then the Hilbert series* $\mathrm{Hilb}(W,t_1,\ldots,t_m)$ *determines* W *up to isomorphism.*

Corollary 12.4.8 *Let* $F_m(R)$ *be the relatively free algebra of the variety generated by a PI-algebra* R*. The Hilbert series* $\mathrm{Hilb}(F_m(R),t_1,\ldots,t_m)$ *is a formal series of Schur functions and if*

$$\mathrm{Hilb}(F_m(R), t_1, \ldots, t_m) = \sum_\lambda k_\lambda(R) S_\lambda(t_1, \ldots, t_m),\ \lambda = (\lambda_1, \ldots, \lambda_m),$$

then

$$F_m(R) \cong \sum_\lambda k_\lambda(R) W_m(\lambda),$$

i.e. the Hilbert series of $F_m(R)$ determines the $GL_m(K)$-module structure of $F_m(R)$.

Proof. By Exercise 12.4.2 (ii) and Theorem 12.4.3, $F_m(R)$ is a direct sum of (maybe infinitely many) irreducible polynomial $G_m(K)$-modules $W_m(\lambda)$, $\lambda = (\lambda_1, \ldots, \lambda_m)$. Now the proof follows from Theorem 12.4.7 applied to $F_m^{(n)}(R)$.

Exercise 12.4.9 Let W be a $GL_m(K)$-submodule of $K\langle V_m\rangle$ and let

$$D_m = \left\{ d(\xi) = d(\xi_1, \ldots, \xi_m) = \sum_{i=1}^m \xi_i e_{ii} \in GL_m(K) \right\}$$

be the diagonal subgroup of $GL_m(K)$. Show that the Hilbert series of W plays the role of the character of D_m and

$$\mathrm{Hilb}(W, \xi_1 t, \ldots, \xi_m t) = \sum_{m \geq 0} \mathrm{tr}_{W^{(n)}}(d(\xi)) t^n,$$

where $\mathrm{tr}_{W^{(n)}}(d(\xi))$ is the trace of $d(\xi)$ acting on the homogeneous component $W^{(n)}$ of degree n of W.

Hint. Use that W is a multigraded vector subspace of $K\langle V_m\rangle$ and the multihomogeneous polynomials are the eigenvectors of $d(\xi)$. For a multihomogeneous polynomial $f(x_1, \ldots, x_m)$ of degree n_i in x_i, the action of $d(\xi)$ gives

$$d(\xi) f(x_1, \ldots, x_m) = \xi_1^{n_1} \ldots \xi_m^{n_m} f(x_1, \ldots, x_m).$$

Exercise 12.4.10 (i) Let $B_m^{(n)}$, $m > 1$, be the subspace of all homogeneous proper polynomials of degree n in $K\langle V_m\rangle$. Show that $B_m^{(n)}$ is a $GL_m(K)$-submodule of $K\langle V_m\rangle$. Find the decomposition of $B_m^{(n)}$ into a sum of irreducible submodules for $n = 2, 3, 4, 5$.

(ii) Find the decomposition of the vector space of homogeneous Lie polynomials of degree $n = 2, 3, 4, 5$ in $K\langle V_m\rangle$, $m > 1$.

Sketch of Solution. (i) Since the elements of $B_m^{(n)}$ are linear combinations of products of commutators, it is easy to see that $B_m^{(n)}$ is $GL_m(K)$-invariant. As graded vector spaces $B_m^{(2)}$ and $B_m^{(3)}$ have respectively bases

$$\{[x_i,x_j]\mid i>j\},\ \{[x_i,x_j,x_k]\mid i>j\leq k\}.$$

We define a correspondence between the basis elements $[x_i,x_j]$, $i>j$, of $B_m^{(2)}$ and the semistandard (1^2)-tableaux with column filled in with j,i. By Theorem 12.4.4 (iii)

$$\mathrm{Hilb}(B_m^{(2)},t_1,\ldots,t_m)=S_{(1^2)}(t_1,\ldots,t_m)$$

and $B_m^{(2)}\cong W_m(1^2)$. For $n=3$ we relate to $[x_i,x_j,x_k]$, $i>j\leq k$ the semistandard $(2,1)$-tableau in Fig. 12.11, which gives $B_m^{(3)}\cong W_m(2,1)$.

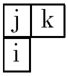


Fig. 12.11. The $(2,1)$-tableau related to $[x_i,x_j,x_k]$, $i>j\leq k$

For $n=4$ we use the basis of $B_m^{(4)}$

$$\{[x_i,x_j,x_k,x_l]\mid i>j\leq k\leq l\}\cup\{[x_i,x_j][x_k,x_l]\mid i>j,\ k>l\}.$$

As for $n=2$ and $n=3$, the Hilbert series of the subspace spanned by $[x_i,x_j,x_k,x_l]$, $i>j\leq k\leq l$, (which is not a $GL_m(K)$-submodule!) is equal to the Schur function $S_\lambda(t_1,\ldots,t_m)$ for $\lambda=(3,1)$ (prove it!). Let

$$W=\mathrm{span}\{[x_i,x_j][x_k,x_l]\mid i>j,\ k>l\}.$$

Since W is a homogeneous polynomial $GL_m(K)$-module of degree 4, its Hilbert series

$$\mathrm{Hilb}(W)=\mathrm{Hilb}(W,t_1,\ldots,t_m)=\sum_{\lambda\vdash 4}k(\lambda)S_\lambda(t_1,\ldots,t_m).$$

is a symmetric polynomial of degree 4. It is sufficient to consider semistandard λ-tableaux of content

$$(n_1,n_2,n_3,n_4)=(n_1,n_2,n_3,n_4,0,\ldots,0),\ n_1\geq n_2\geq n_3\geq n_4.$$

For $(n_1,n_2,n_3,n_4)=(4,0,0,0)$ we obtain that there is only one semistandard tableau and it is for $\lambda=(4)$. Hence

$$0=\mathrm{Hilb}(W,t_1,0,\ldots,0)=k(4),\ k(4)=0.$$

Similarly, for $(n_1,n_2,n_3,n_4)=(3,1,0,0)$, the homogeneous component of W is equal to 0. The only semistandard tableau for $\lambda\neq(4)$, i.e. for $\lambda=(3,1),(2^2),(2,1^2),(1^4)$, is given in Fig. 12.12 and $k(3,1)=0$.

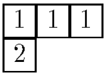
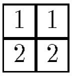
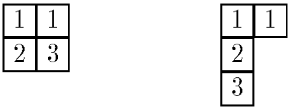


Fig. 12.12. The only semistandard λ-tableau of content $(3,1,0,0)$ for $\lambda \neq (4)$

$$(2,2,0,0):\ [x_2,x_1][x_2,x_1]$$



$$(2,1,1,0):\ [x_2,x_1][x_3,x_1],[x_3,x_1][x_2,x_1]$$



$$(1,1,1,1):\ [x_2,x_1][x_4,x_3],[x_3,x_1][x_4,x_2],[x_4,x_1][x_3,x_2],$$
$$[x_4,x_3][x_2,x_1],[x_4,x_2][x_3,x_1],[x_3,x_2][x_4,x_1]$$

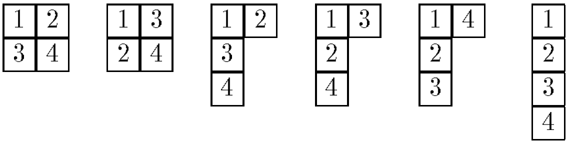


Fig. 12.13. Basis vectors of W and the corresponding tableaux

Counting the basis vectors of W of degree $(2,2,0,0)$ and the semistandard λ-tableaux of content $(2,2,0,0)$, and similarly for $(2,1,1,0)$ and $(1,1,1,1)$, we obtain respectively the polynomials and tableaux in Fig. 12.13. In this way we obtain the linear system

$$\begin{aligned}1 &= k(2^2)\cdot 1 + k(2,1^2)\cdot 0 + k(1^4)\cdot 0\\ 2 &= k(2^2)\cdot 1 + k(2,1^2)\cdot 1 + k(1^4)\cdot 0\\ 6 &= k(2^2)\cdot 2 + k(2,1^2)\cdot 3 + k(1^4)\cdot 1\end{aligned}$$

and its only solution is

$$k(2^2) = k(2,1^2) = k(1^4) = 1.$$

Hence

$$\mathrm{Hilb}(W) = S_{(2^2)} + S_{(2,1^2)} + S_{(1^4)},$$

$$\mathrm{Hilb}(B_m^{(4)}) = S_{(3,1)} + \mathrm{Hilb}(W),$$

$$B_m^{(4)} \cong W_m(3,1) \oplus W_m(2^2) \oplus W_m(2,1^2) \oplus W_m(1^4).$$

Similar calculations give that

$$B_m^{(5)} \cong W_m(4,1) \oplus 2W_m(3,2) \oplus 2W_m(3,1^2) \oplus 2W_m(2^2,1) \oplus 2W_m(2,1^3).$$

(ii) The calculations are as in (i). The answer is

$$L_m^{(2)} = B_m^{(2)} \cong W_m(1^2),\ L_m^{(3)} = B_m^{(3)} \cong W(2,1),$$

$$L_m^{(4)} \cong W_m(3,1) \oplus W_m(2,1^2),$$

$$L_m^{(5)} \cong W_m(4,1) \oplus W_m(3,2) \oplus W_m(3,1^2) \oplus W_m(2^2,1) \oplus W_m(2,1^3).$$

There is a close relation between the irreducible polynomial representations of $GL_m(K)$ and the irreducible representations of the symmetric group S_n. In order to state an analogue of Theorem 12.2.7, we introduce a right action of S_n on $(K\langle V_m\rangle)^{(n)}$ by

$$(x_{i_1}\dots x_{i_n})\sigma^{-1} = x_{i_{\sigma(1)}}\dots x_{i_{\sigma(n)}},\ x_{i_1}\dots x_{i_n} \in (K\langle V_m\rangle)^{(n)},\ \sigma \in S_n.$$

Pay attention that the left action of S_n on P_n (Exercise 12.3.1) is an action on the variables and now S_n acts on the positions of the variables.

Exercise 12.4.11 Show that $(K\langle V_m\rangle)^{(n)}$ is a right S_n-module under the above action.

Hint. Show that $(f\sigma)\tau = f(\sigma\tau)$ for $f \in (K\langle V_m\rangle)^{(n)}$ and $\sigma, \tau \in S_n$. Pay attention that we need σ^{-1} in the definition of the right S_n-action because we consider the permutations as functions, i.e. in $\sigma\tau$ we first apply τ and then σ.

Let $\lambda = (\lambda_1, \dots, \lambda_m)$ be a partition of n in not more than m parts and let $q_1, \dots, q_k$ be the lengths of the columns of the diagram $[\lambda]$ (i.e. $k = \lambda_1$ and $q_j = \lambda_j'$). We denote by $s_\lambda = s_\lambda(x_1, \dots, x_q)$, $q = q_1$, the polynomial of $K\langle V_m\rangle$

$$s_\lambda(x_1, \dots, x_q) = \prod_{j=1}^{k} s_{q_j}(x_1, \dots, x_{q_j}),$$

where $s_p(x_1, \dots, x_p)$ is the standard polynomial.

Theorem 12.4.12 *Let $\lambda = (\lambda_1, \dots, \lambda_m)$ be a partition of n in not more than m parts and let $(K\langle V_m\rangle)^{(n)}$ be the homogeneous component of degree n in $K\langle V_m\rangle$.*

(i) *The element $s_\lambda(x_1, \dots, x_q)$, defined above, generates an irreducible $GL_m(K)$-submodule of $(K\langle V_m\rangle)^{(n)}$ isomorphic to $W_m(\lambda)$.*

(ii) *Every* $W_m(\lambda) \subset (K\langle V_m\rangle)^{(n)}$ *is generated by a nonzero element*

$$w_\lambda(x_1, \ldots, x_q) = s_\lambda(x_1, \ldots, x_q) \sum_{\sigma \in S_n} \alpha_\sigma \sigma,\ \alpha_\sigma \in K.$$

The element $w_\lambda(x_1, \ldots, x_q)$ *is called the highest weight vector of* $W_m(\lambda)$. *It is unique up to a multiplicative constant and is contained in the one-dimensional vector space of the multihomogeneous elements of degree* $(\lambda_1, \ldots, \lambda_m)$ *in* $W_m(\lambda)$.

(iii) *If the* $GL_m(K)$*-submodules* W' *and* W'' *of* $(K\langle V_m\rangle)^{(n)}$ *are isomorphic to* $W_m(\lambda)$ *and have highest weight vectors* w' *and* w'', *respectively, then the mapping* $\phi_\alpha : w' \to \alpha w''$, $0 \neq \alpha \in K$, *can be uniquely extended to a* $GL_m(K)$*-module isomorphism. Every isomorphism* $W' \cong W''$ *is obtained in this way.*

Example 12.4.13 (Compare with Exercise 12.2.10.) Let $m \geq 2$ and let $\lambda = (2,1)$. Applying Theorem 12.4.4 (ii), we use that $d_{(2,1)} = \dim M(2,1) = 2$ and obtain that $W_m(2,1)$ participates in the decomposition of $(K\langle V_m\rangle)^{(3)}$ with multiplicity 2. The lengths of the columns of the Young diagram $[2,1]$ are equal, respectively, to 2 and 1. Hence

$$w' = s_{(2,1)} = s_2(x_1, x_2)s_1(x_1) = [x_1, x_2]x_1 = x_1x_2x_1 - x_2x_1x_1$$

is the highest weight vector of a $GL_m(K)$-submodule $W' \cong W_m(2,1)$ of $(K\langle V_m\rangle)^{(3)}$. For $\sigma = \sigma^{-1} = (13)$

$$w'' = s_{(2,1)}\sigma = x_1x_2x_1 - x_1x_1x_2 = -x_1[x_1, x_2]$$

is a highest weight vector of another submodule $W'' \cong W_m(2,1)$. Since the highest weight vectors in W' and W'' are unique up to multiplicative constants, and w' and w'' are linearly independent, we obtain that $W' \cap W'' = 0$ and $W' \oplus W''$ is a direct summand of $(K\langle V_m\rangle)^{(3)}$. All isomorphisms of the $GL_m(K)$-modules W' and W'' are determined by the mapping

$$\phi_\alpha : w' \to \alpha w'', 0 \neq \alpha \in K.$$

Since every submodule $W \subset (K\langle V_m\rangle)^{(3)}$, $W \cong W_m(2,1)$, is contained in $W' \oplus W''$, the highest weight vector of W is $w = \xi w' + \eta w''$, $(\xi, \eta) \neq (0,0)$.

By Theorem 12.4.4 (ii), the multiplicity of $W_m(\lambda)$ in $(K\langle V_m\rangle)^{(n)}$, is equal to the dimension d_λ of the irreducible S_n-module $M(\lambda)$, $\lambda = (\lambda_1, \ldots, \lambda_m) \vdash n$. Hence every $W \cong W_m(\lambda) \subset (K\langle V_m\rangle)^{(n)}$ is a submodule of the direct sum of d_λ isomorphic copies of $W_m(\lambda)$ and the problem is how to find the highest weight vectors of these d_λ modules.

We fix a partition $\lambda = (\lambda_1, \ldots, \lambda_m)$ of n. Let the columns of the Young diagram $[\lambda]$ be of length $q_1, \ldots, q_k$, $k = \lambda_1$. For a permutation $\sigma \in S_n$ we denote by $T(\sigma)$ the λ-tableau such that the first column of $T(\sigma)$ is filled in consequently from top to bottom with the integers $\sigma(1), \ldots, \sigma(q_1)$, the second column is filled in with $\sigma(q_1+1), \ldots, \sigma(q_1+q_2)$, etc.

Proposition 12.4.14 *Let* $\lambda = (\lambda_1, \ldots, \lambda_m)$ *be a partition of* n *and let* $W_m(\lambda) \subset (K\langle V_m\rangle)^{(n)}$. *The highest weight vector* w_λ *of* $W_m(\lambda)$ *can be expressed uniquely as a linear combination of the polynomials* $w_\sigma = s_\lambda \sigma^{-1}$, *where the* σ*'s are such that the* λ*-tableaux* $T(\sigma)$ *are standard.*

Proof. By Theorem 12.2.12 (i), d_λ is equal to the number of standard λ-tableaux. On the other hand, the homogeneous component of degree $\lambda = (\lambda_1, \ldots, \lambda_m)$ of each of the d_λ copies of $W_m(\lambda)$ is one-dimensional. Hence the λ-homogeneous component of the direct sum is of dimension equal to d_λ. We shall establish the proposition, if we see that the polynomials w_σ, with $T(\sigma)$ standard, are linearly independent. We consider the lexicographic ordering on $K\langle V_m\rangle$ assuming that $x_1 > \ldots > x_m$. If $T(\sigma)$ is a standard λ-tableau, then its entries increase from top to bottom and from left to right. It is easy to see that the leading term of w_σ is $x_1^{\lambda_1} \ldots x_r^{\lambda_r} \sigma^{-1}$ (prove it!). Hence the polynomials w_σ have pairwise different leading terms for different $T(\sigma)$ and are linearly independent.

Example 12.4.15 Let $\lambda = (2,1)$, as in Example 12.4.13. The standard $(2,1)$-tableaux are given in Fig. 12.14 and are obtained for $\sigma_1 = 1$ and $\sigma_2 = (23)$.

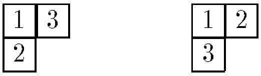


Fig. 12.14. The standard $(2,1)$-tableaux

Hence

$$w_{\sigma_1} = s_{(2,1)} = x_1x_2x_1 - x_2x_1x_1,$$

$$w_{\sigma_2} = s_{(2,1)}(23) = x_1x_1x_2 - x_2x_1x_1,$$

and the leading terms of w_{σ_1} and w_{σ_2} are respectively equal to $x_1x_2x_1$ and $x_1x_1x_2$.

Exercise 12.4.16 Let $k > m$, $k \geq r$, and let $\lambda = (\lambda_1, \ldots, \lambda_r)$ be a partition of n in r parts (i.e. $\lambda_r \neq 0$). Let

$$W_m = K\langle V_m\rangle \cap W_k(\lambda) \subset (K\langle V_k\rangle)^{(n)}.$$

Show that W_m is a $GL_m(K)$-sumbodule of $K\langle V_m\rangle$ such that $W_m = 0$ if $r > m$ and $W_m \cong W_m(\lambda)$ if $r \leq m$.

Hint. Embed $GL_m(K)$ into $GL_k(K)$ fixing the variables $x_{m+1},\dots,x_k$. Use that the Schur functions are expressed in the language of semistandard tableaux and play the role of characters of the irreducible $GL_m(K)$-modules. Show that

$$S_\lambda(t_1,\dots,t_m,0,\dots,0)=0,\ k>m,$$

(no semistandard tableaux of content $(n_1,\dots,n_m,0,\dots,0)$) and

$$S_\lambda(t_1,\dots,t_m,0,\dots,0)=S_\lambda(t_1,\dots,t_m),\ k\leq m.$$

Exercise 12.4.17 If f_1 and f_2 are homogeneous polynomials of degree n in $K\langle V_m\rangle$, prove that the polynomial identity $f_2=0$ is a consequence of $f_1=0$ if and only if f_2 belongs to the $GL_m(K)$-module generated by f_1.

Hint. Let W be the $GL_m(K)$-module generated by f_1. If f_2 belongs to W, then f_2 is a linear combination of $f_1(g(x_1),\dots,g(x_m))$, $g\in GL_m(K)$, and $f_2=0$ is a consequence of $f_1=0$. Now, let $f_2=0$ be a consequence of $f_1=0$. In order to show that $f_2\in W$, it is sufficient to see that $f_1(g(x_1),\dots,g(x_m))\in W$ for every matrix $g=(\alpha_{ij})$ in $M_m(K)$ (not necessarily in $GL_m(K)$). One possible way is the following. The matrix g has a presentation $g=g_1dg_2$, where $g_1,g_2\in GL_m(K)$ and

$$d=(1,\dots,1,0,\dots,0)=\sum_{i=1}^{k}e_{ii}$$

is a diagonal matrix. Extend the action of $GL_m(K)$ on $K\langle V_m\rangle$ to an action of $M_m(K)$ by $g(f)=f(g(x_1),\dots,g(x_m))$ for any $g\in M_m(K)$, $f(x_1,\dots,x_m)\in K\langle V_m\rangle$. Choose a multihomogeneous basis of W. For any basis element $w=w(x_1,\dots,x_m)$ either $d(w)=w$ if w does not depend on $x_{k+1},\dots,x_m$ or $d(w)=0$ otherwise. Hence $d(w)\in W$ for all polynomials $w\in W$. Since W is a $GL_m(K)$-module, we obtain $g_2(f_1)\in W$, hence $d(g_2(f_1))\in W$ and $g(f_1)=g_1(d(g_2(f_1)))\in W$.

Proposition 12.4.18 *Let $m\geq n$, $\lambda\vdash n$ and let $W_m(\lambda)\subset K\langle V_m\rangle$. The set $M=W_m(\lambda)\cap P_n$ of all multilinear elements in $W_m(\lambda)$ is an S_n-submodule of P_n isomorphic to $M(\lambda)$. Every submodule $M(\lambda)$ of P_n can be obtained in this way.*

Proof. Since the $GL_m(K)$-module $W_m(\lambda)$ is irreducible, Exercise 12.4.17 gives that every two nonzero elements of $W_m(\lambda)$ are equivalent as polynomial identities. Hence the nonzero elements in $M=W_m(\lambda)\cap P_n$ are also equivalent. The multilinear consequences of degree n of a multilinear polynomial identity $f(x_1,\dots,x_n)$ belong to the S_n-module generated by f (why?). If M is a direct sum of more than one irreducible submodules $M_1,\dots,M_k$, then the polynomials in M_1 are not consequences of those in M_2. Hence M is irreducible. Let T be the λ-tableau, obtained by filling in the boxes of the first column with $1, \ldots, q_1$, the boxes of the second column with $q_1+1, \ldots, q_1+q_2$, etc. As in Lemma 12.3.8, we can see that the polynomial $\iota(e(T))$ defined there is equivalent to $s_\lambda(x_1, \ldots, x_m)$. Hence, in this special case M is isomorphic to $M(\lambda)$. Now we use that the symmetric group S_n is embedded into $GL_m(K)$ (permuting the first n variables and acting identically on the other $m-n$ variables). If W_1 and W_2 are submodules of $K\langle V_m\rangle$, both isomorphic to $W_m(\lambda)$, then $GL_m(K)$ acts "in the same way" on W_1 and W_2. Hence S_n, as a subgroup of $GL_m(K)$, acts "in the same way" on $M_1 = W_1 \cap P_n$ and $M_2 = W_2 \cap P_n$, and this implies that $M_1 \cong M_2 \cong M(\lambda)$. More precisely, every $GL_m(K)$-module isomorphism $\phi : W_1 \longrightarrow W_2$ is also an isomorphism of multigraded vector spaces. Hence ϕ maps the multilinear elements of W_1 onto the multilinear elements of W_2, $\phi(M_1) = M_2$ and the restriction of ϕ on M_1 is a vector space isomorphism of M_1 to M_2 such $\phi(\sigma(f)) = \sigma(\phi(f))$ for every $f \in M_1$ and every $\sigma \in S_n \subset GL_m(K)$.

Remark 12.4.19 Comparing Exercise 12.2.10 and Example 12.4.13, we see that the highest weight vector of any irreducible $GL_m(K)$-submodule of $K\langle V_m\rangle$ isomorphic to $W_m(2,1)$ is of the form

$$\xi[x_1, x_2]x_1 + \eta x_1[x_1, x_2],\ (0,0) \neq (\xi, \eta) \in K^2,$$

and the expression of this element is "simpler" than the expression of the corresponding multilinear element in $M(2,1) \subset P_3$. In many cases, it is more convenient to use the representation theory of the general linear group instead of the representation theory of the symmetric group. The next theorem shows that it is very easy to translate the results from one of these two languages to the other.

Theorem 12.4.20 (Berele [29], Drensky [71]) *Let R be a PI-algebra and let*

$$\chi_n(R) = \sum_{\lambda \vdash n} m_\lambda(R)\chi_\lambda,\ n = 0, 1, 2, \ldots,$$

be the cocharacter sequence of the T-ideal of R. Then, for any m, the relatively free algebra $F_m(R)$ is isomorphic as a $GL_m(K)$-module to the direct sum

$$\sum_{n \geq 0} \sum_{\lambda \vdash n} m_\lambda(R) W_m(\lambda),$$

with the same multiplicities $m_\lambda(R)$ as in the cocharacter sequence (assuming that $W_m(\lambda) = 0$ if λ is a partition in more than m parts). The Hilbert series of $F_m(R)$ is

$$\mathrm{Hilb}(F_m(R), t_1, \ldots, t_m) = \sum_{n \geq 0} \sum_{\lambda \vdash n} m_\lambda(R) S_\lambda(t_1, \ldots, t_m).$$

On the other hand, if $m \geq n$ *and*

$$F_m^{(n)}(R) \cong \sum_{\lambda \vdash n} n_\lambda(R) W_m(\lambda),$$

for some $n_\lambda(R)$*, then*

$$\chi_n(R) = \sum_{\lambda \vdash n} n_\lambda(R) \chi_\lambda.$$

Proof. Let $k \geq n$ and let

$$F_k^{(n)}(R) \cong \sum_{\lambda \vdash n} n_\lambda(R) W_k(\lambda).$$

By Exercise 12.4.16, for every $m < k$

$$F_m^{(n)}(R) \cong \sum n_\lambda(R) W_m(\lambda),$$

where the summation runs on all partitions $\lambda = (\lambda_1, \ldots, \lambda_m) \vdash n$. Hence, it is sufficient to show that $P_n(R)$ and $F_m^{(n)}(R)$ have "the same" module structure for $m \geq n$ (i.e. the same multiplicities respectively of $M(\lambda)$ and $W_m(\lambda)$). The decomposition of the n-th cocharacter of $T(R)$ is equivalent to the decomposition of $P_n(R)$

$$P_n(R) = \sum_{\lambda \vdash n} m_\lambda(R) M(\lambda).$$

Since $m \geq n$ and λ is a partition of n, every irreducible submodule $W_m(\lambda)$ of $F_m^{(n)}(R)$ intersects with some irreducible submodule $M(\lambda)$ of $P_n(R)$. The direct sum of $m_\lambda(R)$ copies of $M(\lambda)$ in $P_n(R)$ generates a $GL_m(K)$-module which is a direct sum of $m_\lambda(R)$ copies of $W_m(\lambda)$ in $F_m^{(n)}(R)$. In is easy to see that this means

$$F_m^{(n)}(R) = \sum_{\lambda \vdash n} m_\lambda(R) W_m(\lambda),$$

and this competes the proof.

Remark 12.4.21 An analogue of Theorem 12.4.20 holds also for other varieties of linear algebras over a field K of characteristic 0 and in the more general case when we consider factor algebras of the absolutely free (nonassociative) algebra $K\{X\}$ modulo ideals U invariant under substitutions of linear combinations of the variables, i.e. when $f(x_1, \ldots, x_n) \in U$ implies

$$f(\sum_{i=1}^{k_1} \alpha_{i1} x_i, \ldots, \sum_{i=1}^{k_n} \alpha_{in} x_i) \in U,\ \alpha_{ij} \in K.$$

Example 12.4.22 Theorem 12.4.20 and Remark 12.4.21 allow to determine the S_n-module structure of PL_n and Γ_n for $n \leq 4$ (see Exercises 12.3.2 and 12.3.3) as a direct consequence of Exercise 12.4.10. For example, the $GL_m(K)$-module decomposition

$$B_m^{(4)} \cong W_m(3,1) \oplus W_m(2^2) \oplus W_m(2,1^2) \oplus W_m(1^4)$$

is equivalent to the S_4-module decomposition

$$\Gamma_4 \cong M(3,1) \oplus M(2^2) \oplus M(2,1^2) \oplus M(1^4).$$

Now we shall show how to decompose $B_m^{(5)}$ using highest weight vectors. We calculate (using e.g. the hook formula of Theorem 12.32) the dimensions of the irreducible S_5-modules

$$\dim M(4,1) = \dim M(2,1^3) = 4,$$

$$\dim M(3,2) = \dim M(2^2,1) = 5,\ \dim M(3,1^2) = 6.$$

Since

$$\dim \Gamma_5 = 44 = \dim M(4,1) + 2\dim M(3,2)+$$

$$+2\dim M(3,1^2) + 2\dim M(2^2,1) + 2\dim M(2,1^3),$$

it is sufficient to find in $B_m^{(5)}$ a highest weight vector $w_{(4,1)}$ and two linearly independent highest weight vectors w'_λ and w''_λ for every $\lambda = (3,2)$, $(3,1^2)$, $(2^2,1)$, $(2,1^3)$. This would guarantee that the $GL_m(K)$-module

$$W_m(4,1) \oplus 2W_m(3,2) \oplus 2W_m(3,1^2) \oplus 2W_m(2^2,1) \oplus 2W_m(2,1^3)$$

is a submodule of $B_m^{(5)}$, hence

$$M(4,1) \oplus 2M(3,2) \oplus 2M(3,1^2) \oplus 2M(2^2,1) \oplus 2M(2,1^3)$$

would be an S_5-submodule of Γ_5. Since the dimension of this S_5-submodule is equal to the dimension of Γ_5, we obtain that it coincides with Γ_5. The element

$$s_{(4,1)} = s_2(x_1,x_2)x_1^3$$

is a highest weight vector of $(K\langle V_m\rangle)^{(5)}$. We define a $GL_m(K)$-module homomorphism $\phi : (K\langle V_m\rangle)^{(5)} \longrightarrow B_m^{(5)}$ by

$$\phi : x_{i_1}\dots x_{i_5} \to [x_{i_1},\dots,x_{i_5}]$$

and obtain that

$$\phi(s_{(4,1)}) = w_{(4,1)} = -2[x_2,x_1,x_1,x_1,x_1] \neq 0$$

in $B_m^{(5)}$. Hence $W_m(4,1) \subset B_m^{(5)}$. Similarly, for $\lambda = (3,2)$ we define $GL_m(K)$-module homomorphisms $\phi_i : (K\langle V_m\rangle)^{(5)} \longrightarrow B_m^{(5)}$, $i = 1,2$, by

$$\phi_1 : x_{i_1}\dots x_{i_5} \to [x_{i_1}, x_{i_2}, x_{i_5}][x_{i_3}, x_{i_4}]$$

$$\phi_2 : x_{i_1}\dots x_{i_5} \to [x_{i_1}, x_{i_2}][x_{i_3}, x_{i_4}, x_{i_5}]$$

and the polynomials

$$\phi_1(s_{(3,2)}) = 4[x_2, x_1, x_1][x_2, x_1],\ \phi_2(s_{(3,2)}) = 4[x_2, x_1][x_2, x_1, x_1]$$

are linearly independent highest weight vectors. Hence $2W_m(3,2) \subset B_m^{(5)}$. The considerations for the other λ's are analogous.

12.5 Proper Polynomial Identities

In Section 4.3 we have established relations between the ordinary and the proper polynomial identities. It is naturally to expect that the information about the representations of S_n and $GL_m(K)$ related with the polynomial identities of a PI-algebra R can be expressed in terms of the corresponding representations related with the proper identities. It turns out that such relations do exist and they require deeper results on representation theory of $GL_m(K)$ than these used in Section 12.4. In particular, we need some information on tensor products of $GL_m(K)$-modules.

Exercise 12.5.1 Show that the tensor product $W_1 \otimes W_2$ of two polynomial $GL_m(K)$-modules W_1 and W_2 is a polynomial module again.

Hint. Take bases $\{w_i' \mid i \in I\}$ and $\{w_j'' \mid j \in J\}$ of W_1 and W_2, respectively. Use that for $g \in GL_m(K)$

$$g(w_i') = \sum_{p\in I} \alpha_{pi} w_p',\ g(w_j'') = \sum_{q\in J} \beta_{qj} w_q'',$$

where α_{pi} and β_{qj} are polynomials of the entries of g, implies that

$$g(w_i' \otimes w_j'') = \sum \alpha_{pi}\beta_{qj}(w_p' \otimes w_q'').$$

Derive from here that the entries of the matrix corresponding to the action of g on $W_1 \otimes W_2$ (with respect to the basis $\{w_i' \otimes w_j'' \mid i \in I, j \in J\}$) are again polynomials of the entries of g.

The decomposition into a sum of irreducible $GL_m(K)$-submodules of the tensor product $W_1 \otimes W_2$ of two irreducible polynomial $GL_m(K)$-modules W_1 and W_2 can be obtained by the combinatorial *Littlewood-Richardson rule* (see e.g. the book by Macdonald [175]). There is a simplified version of this rule which gives the decomposition when one of the modules is isomorphic to $W_m(n)$ or $W_m(1^n)$.

Theorem 12.5.2 (The Young Rule) *The tensor products of the irreducible $GL_m(K)$-module $W_m(\lambda_1,\ldots,\lambda_m)$ with $W_m(q)$ and $W_m(1^q)$ (in the latter case $q\leq m$) are decomposed into the following sums of irreducible components:*

$$W_m(\lambda_1,\ldots,\lambda_m)\otimes W_m(q)\cong W_m(\lambda_1+p_1,\ldots,\lambda_m+p_m),$$

where the summation is over all nonnegative integers $p_1,\ldots,p_m$ such that $p_1+\ldots+p_m=q$ and $\lambda_i+p_i\leq\lambda_{i-1}$, $i=2,\ldots,m$;

$$W_m(\lambda_1,\ldots,\lambda_m)\otimes W_m(1^q)\cong W_m(\lambda_1+\varepsilon_1,\ldots,\lambda_m+\varepsilon_m),$$

where the summation is over all $\varepsilon_i=0,1$, such that $\varepsilon_1+\ldots+\varepsilon_m=q$ and $\lambda_i+\varepsilon_i\leq\lambda_{i-1}+\varepsilon_{i-1}$, $i=2,\ldots,m$.

In other words, $W_m(\lambda)\otimes W_m(q)$ and $W_m(\lambda)\otimes W_m(1^q)$ are direct sums of pairwise nonisomorphic irreducible $GL_m(K)$-modules. The diagrams $[\mu]$ corresponding to the irreducible components of $W_m(\lambda)\otimes W_m(q)$ are obtained from the diagram $[\lambda]$ by adding q boxes in such a way that no two new boxes are in the same column of $[\mu]$. For $W_m(\lambda)\otimes W_m(1^q)$ the new q boxes are not allowed to be in the same row. When $q=1$ this is the well known *Branching Theorem.*

Example 12.5.3 Let $\lambda=(2,1)$ and $q=2$. Then (see Fig. 12.15 and 12.16), for $m\geq 4$,

$$W_m(2,1)\otimes W_m(2)\cong W_m(4,1)\oplus W_m(3,2)\oplus W_m(3,1^2)\oplus W_m(2^2,1),$$

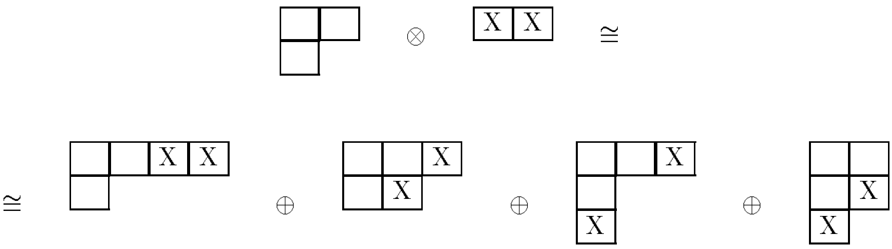


Fig. 12.15. The decomposition of $W_m(2,1)\otimes W_m(2)$

$$W_m(2,1)\otimes W_m(1^2)\cong W_m(3,2)\oplus W_m(3,1^2)\oplus W_m(2^2,1)\oplus W_m(2,1^3).$$

The following theorem gives the relationship between the ordinary and proper cocharacters of a PI-algebra. As in Chapters 4 and 5, it allows to simplify the calculations.

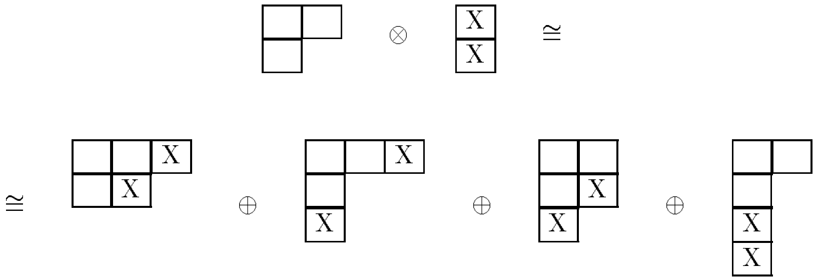


Fig. 12.16. The decomposition of $W_m(2,1) \otimes W_m(1^2)$

Theorem 12.5.4 (Drensky [74]) *Let R be a unitary PI-algebra and let*

$$\chi_n(R) = \sum_{\lambda \vdash n} m_\lambda(R)\chi_\lambda,\ n = 0,1,2,\ldots,$$

$$\eta_p(R) = \chi_{\Gamma_p(R)} = \sum_{\nu \vdash p} k_\nu(R)\chi_\nu,\ p = 0,1,2,\ldots,$$

be, respectively, the ordinary and proper cocharacter sequences of the T-ideal $T(R)$. Then the multiplicities $m_\lambda(R)$ and $k_\nu(R)$ are related by

$$m_\lambda(R) = \sum k_\nu(R),$$

where for $\lambda = (\lambda_1,\ldots,\lambda_n)$ the summation runs on all partitions $\nu = (\nu_1,\ldots,\nu_n)$ such that

$$\lambda_1 \geq \nu_1 \geq \lambda_2 \geq \nu_2 \geq \ldots \geq \lambda_n \geq \nu_n.$$

Proof. By Theorem 4.3.12 (i), the Hilbert series of the relatively free algebra

$$F_m(R) = K\langle x_1,\ldots,x_m\rangle/(K\langle x_1,\ldots,x_m\rangle \cap T(R))$$

and its proper elements $B_m(R)$ are related by

$$\mathrm{Hilb}(F_m(R),t_1,\ldots,t_m) = \mathrm{Hilb}(B_m(R),t_1,\ldots,t_m)\prod_{i=1}^{m}\frac{1}{1-t_i}.$$

The graded vector spaces with finite dimensional homogeneous components are determined up to isomorphism by their Hilbert series. Hence, as a multigraded vector space, $F_m(R)$ is isomorphic to the tensor product of $B_m(R)$ and the polynomial algebra $K[V_m]$. By Corollary 12.4.8, the Hilbert series of $F_m(R)$ determines the $GL_m(K)$-module structure of $F_m(R)$. Hence, as $GL_m(K)$-modules,

$$F_m(R) \cong B_m(R) \otimes K[x_1, \dots, x_m].$$

On the other hand, since the Hilbert series of the vector space of all homogeneous polynomials of degree n is equal to the complete symmetric function

$$h_n(t_1, \dots, t_m) = \sum t_1^{n_1} \dots t_m^{n_m},\ n_1 + \dots + n_m = n,$$

Exercise 12.4.6 gives that as $GL_m(K)$-modules

$$K[x_1, \dots, x_m] \cong \sum_{n \geq 0} W_m(n).$$

By the equivalence between representations of S_n and $GL_m(K)$ (Theorem 12.4.20 and Remark 12.4.21),

$$F_m(R) \cong \sum_{\lambda} m_\lambda(R) W_m(\lambda) \cong \sum_{\nu} \sum_{p \geq 0} k_\nu(R) W_m(\nu) \otimes W_m(p).$$

Now we apply the Young rule of Theorem 12.5.2. The module $W_m(\lambda)$ participates in the decomposition of $W_m(\nu) \otimes W_m(n)$ if and only if there are no boxes in the same columns in the set difference of Young diagrams $[\lambda] \setminus [\nu]$, i.e. if the lengths of the rows of $[\lambda] = [\lambda_1, \dots, \lambda_n]$ and $[\nu] = [\nu_1, \dots, \nu_n]$ satisfy the inequalities

$$\lambda_1 \geq \nu_1 \geq \lambda_2 \geq \nu_2 \geq \dots \geq \lambda_n \geq \nu_n.$$

Using again the equivalence between representations of S_n and $GL_m(K)$, we obtain the desired decomposition.

Now we shall apply Theorem 12.5.4 to calculate the cocharacters of some important algebras already considered in the previous chapters. We start with the cocharacters of the Grassmann algebra. The original theorem is due to Krakowski and Regev [155] and is based on representation theory of the symmetric group. An alternative exposition can be found in the paper by Ananin and Kemer [14].

Theorem 12.5.5 (Krakowski and Regev [155]) *Let E be the Grassmann algebra of an infinite dimensional vector space. Then the cocharacter sequence of the polynomial identities of E is*

$$\chi_n(E) = \sum_{k=0}^{n-1} \chi_{(n-k, 1^k)},$$

i.e. the irreducible S_n-submodules of $P_n(E)$ are with multiplicity 1 *and correspond to Young diagrams lying in a hook with height of the arm and wide of the leg equal to* 1.

Proof. We apply Theorem 5.1.2 (iii) (and its proof). The Hilbert series of $B_m(E)$ is

$$\mathrm{Hilb}(B_m(E),t_1,\ldots,t_m)=\sum_{k\geq 0}e_{2k}(t_1,\ldots,t_m),$$

where $e_p(t_1,\ldots,t_m)$ is the elementary symmetric polynomial of degree p. By Exercise 12.4.6,

$$\mathrm{Hilb}(B_m(E),t_1,\ldots,t_m)=\sum_{k\geq 0}S_{(1^{2k})}(t_1,\ldots,t_m),$$

and $B_m(E)$ is a direct sum of the $GL_m(K)$-modules $W_m(1^{2k})$, $k=0,1,2,\ldots$ Hence, by Theorem 12.5.4, $\chi_n(E)$ is a sum of the irreducible S_n-characters χ_λ, where the diagram $[\lambda]$ is obtained from the diagram $[\nu]=[1^{2k}]$ by adding several boxes to the first row and, eventually, building a new row of one box (with odd number of the row). Hence $\lambda=(a,1^b)\vdash n$, where $a+b=n$.

The next theorem describes the $GL_m(K)$-module structure of the proper elements of the relatively free algebra $F_m(U_k(K))$ of the variety of associative algebras generated by the algebra of $k\times k$ upper triangular matrices.

Theorem 12.5.6 (Drensky and Kasparian [91]) *Let $U_k(K)$ be the algebra of $k\times k$ upper triangular matrices. Then*

$$B_m(U_k(K))\cong\sum_{r=0}^{k-1}\sum_{p_i\geq 2}W_m(p_1-1,1)\otimes\ldots\otimes W_m(p_r-1,1).$$

Proof. First, let $k=2$. By Theorem 5.2.1, $B_m^{(p)}(U_2(K))$, $p>0$, has a basis

$$[x_{i_1},x_{i_2},\ldots,x_{i_p}],\ i_1>i_2\leq i_3\leq\ldots\leq i_p.$$

We define a 1-1 correspondence between the basis elements of $B_m^{(p)}(U_2(K))$ and the semistandard $(p-1,1)$-tableaux by

$$[x_{i_1},x_{i_2},\ldots,x_{i_p}]\qquad\longrightarrow\qquad\begin{array}{|c|c|c|c|}\hline i_2 & i_3 & \ldots & i_p\\\hline i_1 & \multicolumn{3}{c}{}\\\cline{1-1}\end{array}$$

Clearly, the image of a basis element of degree $\alpha=(\alpha_1,\ldots,\alpha_m)$ is a semistandard tableau of the same content α. Hence the Hilbert series of $B_m^{(p)}(U_2(K))$ and the Schur function $S_{(p-1,1)}(t_1,\ldots,t_m)$ are equal. Bearing in mind that $B_m^{(0)}(U_2(K))=K$ and the Hilbert series determines the $GL_m(K)$-module structure, we obtain that

$$B_m(U_2(K))\cong K\oplus\sum_{p\geq 2}W_m(p-1,1).$$

Applying again Theorem 5.2.1, we obtain that, as a graded vector space, $B_m(U_k(K))$, $k>2$, is isomorphic to

$$B_m(U_{k-1}(K)) \oplus \sum_{p_i \geq 2} W_m(p_1-1,1) \otimes \ldots \otimes W_m(p_{k-1}-1,1)$$

and, by induction, this completes the proof of the theorem.

Remark 12.5.7 In order to obtain the cocharacter sequence of $T(U_k(K))$, we need to know the decomposition of the tensor products $W_m(\lambda) \otimes W_m(p-1,1)$. For $p=2$ we can apply the Young rule of Theorem 12.5.2. For $p>2$ we need the Littlewood-Richardson rule.

Exercise 12.5.8 Show, that as a $GL_m(K)$-module, $B_m \subset K\langle V_m \rangle$ is isomorphic to the tensor product

$$\sum_{r \geq 0} \sum_{p_i \geq 2} W_m(p_1-1,1) \otimes \ldots \otimes W_m(p_r-1,1).$$

Hint. Apply Theorem 12.5.6. Use that the minimal degree of the polynomial identities of $U_k(K)$ is equal to $2k$ and hence $B_m^{(n)} \cong B_m^{(n)}(U_k(K))$ for $2k>n$.

Exercise 12.5.9 Using Example 12.5.3 and Exercise 12.5.8, decompose once again $B_m^{(n)}$ for $n=4$ and $n=5$.

We have paid so much attention to the decomposition of the $GL_m(K)$-module B_m of the proper polynomials in the free algebra $K\langle V_m \rangle$, because the knowledge of the decomposition of $B_m^{(n)}$ for small n is very useful for the description of the polynomial identities of PI-algebras satisfying identities of low degree. It also helps to make conjectures for the cocharacters of T-ideals.

Exercise 12.5.10 Compute the S_n-cocharacter of the 2×2 matrix algebra $M_2(K)$ for $n=5$.

Sketch of Solution. We use the decomposition of $B_m^{(n)}$ for $n \leq 5$, obtained in Exercise 12.4.10 and Example 12.4.22. Let $m \geq 5$. Since the only polynomial identity of degree ≤ 4 for $M_2(K)$ is the standard identity s_4, which generates the $GL_m(K)$-submodule $W_m(1^4)$ of $B_m^{(4)}$, we obtain that $B_m^{(n)}(M_2(K)) = B_m^{(n)}$ for $n \leq 3$,

$$B_m^{(0)}(M_2(K)) = W_m(0) = K, \; B_m^{(1)}(M_2(K)) = 0,$$

$$B_m^{(2)}(M_2(K)) = W_m(1^2), \; B_m^{(3)}(M_2(K)) = W_m(2,1),$$

$$B_m^{(4)}(M_2(K)) \cong B_m^{(4)}/W_m(1^4) \cong W_m(3,1) \oplus W_m(2^2) \oplus W_m(2,1^2).$$

Now we choose the following highest weight vectors of the irreducible components $W_m(\lambda)$ of $B_m^{(5)}$:

$\lambda = (4,1)$:

$$w_{(4,1)} = [x_2, x_1, x_1, x_1, x_1],$$

$\lambda = (3, 2)$:

$$w'_{(3,2)} = [[x_2, x_1, x_1], [x_2, x_1]],\ w''_{(3,2)} = [x_2, x_1, x_1] \circ [x_2, x_1],$$

$\lambda = (3, 1^2)$:

$$w'_{(3,1^2)} = \sum_{\sigma \in S_3} (\text{sign}\sigma)[[x_{\sigma(1)}, x_1, x_1], [x_{\sigma(2)}, x_{\sigma(3)}]],$$

$$w''_{(3,1^2)} = \sum_{\sigma \in S_3} [x_{\sigma(1)}, x_1, x_1] \circ [x_{\sigma(2)}, x_{\sigma(3)}],$$

$\lambda = (2^2, 1)$:

$$w'_{(2^2,1)} = \sum_{\sigma \in S_3} (\text{sign}\sigma)[[x_2, x_1, x_{\sigma(1)}], [x_{\sigma(2)}, x_{\sigma(3)}]],$$

$$w''_{(2^2,1)} = \sum_{\sigma \in S_3} [x_2, x_1, x_{\sigma(1)}] \circ [x_{\sigma(2)}, x_{\sigma(3)}],$$

$\lambda = (2, 1^3)$:

$$w'_{(2,1^3)} = \sum_{\sigma \in S_4} (\text{sign}\sigma)[[x_{\sigma(1)}, x_1, x_{\sigma(2)}], [x_{\sigma(3)}, x_{\sigma(4)}]],$$

$$w''_{(2,1^3)} = \sum_{\sigma \in S_4} (\text{sign}\sigma)[x_{\sigma(1)}, x_1, x_{\sigma(2)}] \circ [x_{\sigma(3)}, x_{\sigma(4)}].$$

Replacing x_1 by a diagonal generic 2×2 matrix and $x_2, \ldots, x_m$ by arbitrary generic matrices we see that

$$w_{(4,1)} \neq 0,\ w'_{(3,2)} \neq 0,\ w''_{(3,1^2)} \neq 0,\ w'_{(2^2,1)} \neq 0$$

in $M_2(K)$ and

$$w''_{(3,2)} = w'_{(3,1^2)} = w''_{(2^2,1)} = w'_{(2,1^3)} = w'_{(2,1^3)} = 0$$

are polynomial identities for $M_2(K)$. Hence (why?)

$$B_m^{(5)}(M_2(K)) \cong W_m(4,1) \oplus W_m(3,2) \oplus W_m(3,1^2) \oplus W_m(2^2,1).$$

Applying Theorem 12.5.4, we obtain that

$$\chi_5(M_2(K)) = \chi_{(5)} + 4\chi_{(4,1)} + 4\chi_{(3,2)} + 5\chi_{(3,1^2)} + 4\chi_{(2^2,1)} + \chi_{(2,1^3)}.$$

Exercise 12.5.11 (Partial case of a result of Popov [209]) Compute the S_n-cocharacter of the tensor square $E \otimes E$ of the Grassmann algebra E for $n = 5$.

Hint. Use Exercise 2.1.6. In the notation of Exercise 12.5.10, show that the polynomials $w'_{(3,1^2)}$, $w'_{(2^2,1)}$, $w'_{(2,1^3)}$ are consequences of the centre-by-metabelian identity. Prove that $w''_{(3,2)}$ is also a polynomial identity for $E\otimes E$. Show that neither the other highest weight vectors nor the highest weight vectors of $B_m^{(4)}$ vanish on $E\otimes E$. Hence

$$B_m^{(n)}(E\otimes E)=B_m^{(n)},\ n\leq 4,$$

$$B_m^{(5)}(E\otimes E)\cong W_m(4,1)\oplus W_m(3,2)\oplus W_m(3,1^2)\oplus W_m(2^2,1)\oplus W_m(2,1^3).$$

Finally, apply Theorem 12.5.4.

Exercise 12.5.12 Prove the following theorem of Regev [224]. The algebra R satisfies the Capelli identity in k skew-symmetric variables if and only if the multiplicities $m_\lambda(R)$ in the cocharacter sequence of R are equal to 0 for all $\lambda=(\lambda_1,\ldots,\lambda_n)$ with $\lambda_k\neq 0$.

Hint. Use that for $\lambda_k\neq 0$, the highest weight vectors w_λ of $W_m(\lambda)\subset K\langle V_m\rangle$ contain sums with $\geq k$ skew-symmetric variables, i.e. the Capelli identity implies $w_\lambda=0$ in R and $m_\lambda(R)=0$. On the other hand, let all multiplicities $m_\lambda(R)$ be equal to 0 for $\lambda_k\neq 0$. We generate a $GL_m(K)$-submodule of $K\langle V_m\rangle$ by the Capelli polynomial. It is equal to 0 for $m<k$, and, by Exercise 12.4.16, for $m\geq k$, it decomposes as a direct sum of $W_m(\lambda)$ with $\lambda_k\neq 0$. Hence the Capelli polynomial belongs to $T(R)$.

Exercise 12.5.13 Let R be a PI-algebra and let $\dim R=p$. Prove, that in the cocharacter sequence of the T-ideal $T(R)$

$$\chi_n(R)=\sum_{\lambda\vdash n}m_\lambda(R)\chi_\lambda,\ n=0,1,2,\ldots,$$

$m_\lambda(R)=0$, if $\lambda_{p+1}\neq 0$ in $\lambda=(\lambda_1,\ldots,\lambda_n)$.

Hint. Use that R satisfies the Capelli identity in $p+1$ skew-symmetric variables and apply Exercise 12.5.12.

Exercise 12.5.14 Let R be a unitary PI-algebra with centre C and let

$$\dim R=p,\ \dim C=q.$$

Prove, that in the ordinary and proper cocharacter sequences of the T-ideal $T(R)$

$$\chi_n(R)=\sum_{\lambda\vdash n}m_\lambda(R)\chi_\lambda,\ n=0,1,2,\ldots,$$

$$\eta_n(R)=\chi_{\Gamma_n(R)}=\sum_{\nu\vdash n}k_\nu(R)\chi_\nu,\ n=0,1,2,\ldots,$$

$m_\lambda(R) = k_\nu(R) = 0$, if $\lambda_{p-q+2} \neq 0$ in $\lambda = (\lambda_1, \ldots, \lambda_n)$ and $\nu_{p-q+1} \neq 0$ in $\nu = (\nu_1, \ldots, \nu_n)$.

Hint. For the proper cocharacters use that every proper polynomial with $p - q + 1$ skew-symmetries vanishes on R. For the ordinary cocharacters apply Theorem 12.5.4.

12.6 Polynomial Identities of Matrices

Since the relatively free algebra $F_m(M_k(K))$ is isomorphic to the algebra generated by m generic $k \times k$ matrices, invariant theory of matrices was successfully involved by Procesi [214] and Razmyslov [219] in the study of the T-ideal $T(M_k(K))$ of the polynomial identities of $k \times k$ matrices over a field of characteristic 0. A survey can be found in the book by Formanek [108].

Up till now the results on $T(M_k(K))$, $k > 2$, are far from their final form and only the structure of the identities of the 2×2 matrices is well known. In a series of papers (see [226] for a survey) Regev obtained the asymptotic behaviour of $c_n(M_2(K))$ and $\chi_n(M_2(K))$; the explicit expression of $\chi_n(M_2(K))$ was found by Formanek [105] and the author [74]; Procesi [215] computed the codimension sequence of $M_2(K)$. Formanek, Halpin and Li [109] for $m = 2$ and Procesi [215], Formanek [105] and Drensky [74] in the general case computed the Hilbert series of $F_m(M_2(K))$.

One of the possible ways to study $T(M_2(K))$ is via the generic trace algebra generated by m generic 2×2 matrices together with all the traces of the generic matrix algebra $F_m(M_2(K))$ and the approach of Regev, Formanek and Procesi follows this way. For a survey on the 2×2 generic trace algebra see the book by Le Bruyn [169]. Till the end of the chapter we give as an application of the method of representation theory of $GL_m(K)$ an exposition of the quantitative results for $T(M_2(K))$ without any invariant theory. The method works also for other PI-algebras and we refer to the survey [77] for further applications.

Exercise 12.6.1 Let

$$a_1 = e_{11} - e_{22},\ a_2 = e_{12} + e_{21},\ a_3 = e_{12} - e_{21} \in M_2(K)$$

and let $a_{i_1} \ldots a_{i_n}$ be a product which contains n_1 times a_1, n_2 times a_2, n_3 times a_3, $n_1 + n_2 + n_3 = n$. Let $\varepsilon_i = 0, 1$, $\varepsilon_i \equiv n_i \pmod 2$, $i = 1, 2, 3$. Show that

$$a_{i_1} \ldots a_{i_n} = \pm a_1^{\varepsilon_1} a_2^{\varepsilon_2} a_3^{\varepsilon_3}.$$

Hint. Show that

$$a_1a_2 = -a_2a_1 = a_3,\ a_2a_3 = -a_3a_2 = -a_1,\ a_3a_1 = -a_1a_3 = -a_2,$$

$$a_1^2 = a_2^2 = e,\ a_3^2 = -e.$$

The equations of the first line give that we can rearrange the elements a_{i_j} and the equations of the second line give the desired result.

The following theorem of Drensky [71, 78] gives a simple test verifying whether a highest weight vector in B_m is a polynomial identity for $M_2(K)$.

Theorem 12.6.2 *Let $w_\mu(x_1,\ldots,x_m)$, $\mu = (\mu_1,\ldots,\mu_m)$, be a highest weight vector of the submodule $W_m(\mu)$ of the $GL_m(K)$-module B_m of the proper polynomials in $K\langle V_m\rangle$. If $\mu_4 \neq 0$, then $w_\mu(x_1,\ldots,x_m)$ is a polynomial identity for $M_2(K)$. When $\mu_4 = 0$, then $w_\mu(x_1,x_2,x_3)$ is a polynomial identity for $M_2(K)$ if and only if $w_\mu(a_1,a_2,a_3) = 0$, where the matrices a_1,a_2,a_3 are defined by*

$$a_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},\ a_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},\ a_3 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Proof. By Exercise 12.1.10 (i), the centre of $M_2(K)$ consists of all scalar matrices and hence is one-dimensional. Since $\dim M_2(K) = 4$, Exercise 12.5.14 gives that the $GL_m(K)$-module $B_m(M_2(K))$ is decomposed into a sum of irreducible $GL_m(K)$-modules $W_m(\mu_1,\mu_2,\mu_3)$ and this implies that $w_\mu \in W_m(\mu)$ is a polynomial identity for $M_2(K)$ if $\mu_4 \neq 0$. Now, let $\mu_4 = 0$. By Exercise 7.2.2, $w_\mu(x_1,x_2,x_3) \in T(M_2(K))$ if and only if $w_\mu(y_1,y_2,y_3) = 0$ for the generic matrices y_1,y_2,y_3. By Exercise 7.2.7 we may assume that y_1 is a diagonal matrix and y_2 is symmetric. The matrices e,a_1,a_2,a_3 form a basis of $M_2(K)$. We rewrite y_1,y_2,y_3 as their linear combinations:

$$y_1 = \phi_{01}e + \xi a_1,\ y_2 = \phi_{02}e + \phi_{12}a_1 + \eta a_2,\ y_3 = \phi_{03}e + \phi_{13}a_1 + \phi_{23}a_2 + \zeta a_3,$$

where ϕ_{ij},ξ,η,ζ are commuting variables. All elements of $w_\mu(y_1,y_2,y_3)$ are in commutators only and we may assume that $\phi_{0i} = 0$. The polynomial $w_\mu(x_1,x_2,x_3)$ is multihomogeneous of degree μ_i in x_i. By the skew-symmetry of $w_\mu(x_1,x_2,x_3)$ described in Theorem 12.4.12 (ii), we obtain that

$$w_\mu(y_1,y_2,y_3) = w_\mu(\xi a_1,\eta a_2,\zeta a_3) = \xi^{\mu_1}\eta^{\mu_2}\zeta^{\mu_3}w_\mu(a_1,a_2,a_3)$$

and $w_\mu(y_1,y_2,y_3) = 0$ if and only if $w_\mu(a_1,a_2,a_3) = 0$.

Remark 12.6.3 The matrices a_1,a_2,a_3 in Theorem 12.6.2 can be replaced by the matrices

$$b_1 = -\frac{1}{2}(e_{11} - e_{22})\sqrt{-1},\ b_2 = \frac{1}{2}(e_{12} + e_{21})\sqrt{-1},\ b_3 = \frac{1}{2}(e_{12} - e_{21}).$$

The table of multiplication of these matrices is the following

$$b_1b_2=-b_2b_1=\frac{b_3}{2},\ b_2b_3=-b_3b_2=-\frac{b_1}{2},\ b_3b_1=-b_1b_3=-\frac{b_2}{2},$$

$$[b_1,b_2]=b_3,\ [b_2,b_3]=b_1,\ [b_3,b_1]=b_2,\ b_1^2=b_2^2=b_3^2=-\frac{e}{4},$$

and this will simplify the computations in the next theorem which describes the $GL_m(K)$-module structure of the proper polynomials in $F_m(M_2(K))$.

Theorem 12.6.4 (Drensky [71]) *The $GL_m(K)$-module of the proper polynomials in the relatively free algebra $F_m(M_2(K))$ satisfies*

$$B_m(M_2(K))=B_m/(B_m\cap T(M_2(K)))\cong\sum W_m(\mu_1,\mu_2,\mu_3),$$

where the summation is over all partitions $\mu=(\mu_1,\mu_2,\mu_3)$ different from $\mu=(1^3)$ and $\mu=(n)$, $n\geq 1$.

Proof. Let

$$B_m(M_2(K))\cong\sum k_\mu(M_2(K))W_m(\mu).$$

By Theorem 12.6.2, $k_\mu(M_2(K))=0$, if $\mu=(\mu_1,\ldots,\mu_m)$ and $\mu_4\neq 0$. Now, let $\mu=(\mu_1,\mu_2,\mu_3)$ and let $w'_\mu(x_1,x_2,x_3)$ and $w''_\mu(x_1,x_2,x_3)$ be highest weight vectors of two isomorphic $GL_m(K)$-submodules $W_m(\mu)$ of $B_m\subset K\langle V_m\rangle$ and let $w'_\mu\notin B_m\cap T(M_2(K))$. Then, by Exercise 12.6.1, (w'_μ and w''_μ are multihomogeneous), there exist constants α' and α'' in K such that

$$w'_\mu(a_1,a_2,a_3)=\alpha' a_1^{\varepsilon_1}a_2^{\varepsilon_2}a_3^{\varepsilon_3},\ w''_\mu(a_1,a_2,a_3)=\alpha'' a_1^{\varepsilon_1}a_2^{\varepsilon_2}a_3^{\varepsilon_3},$$

where $\varepsilon_i=0,1$, $\varepsilon_i\equiv\mu_i \pmod 2$, $i=1,2,3$. By Theorem 12.6.2, $\alpha'\neq 0$. On the other hand,

$$w_\mu(x_1,x_2,x_3)=\alpha'' w'_\mu(x_1,x_2,x_3)-\alpha' w''_\mu(x_1,x_2,x_3)\in B_m$$

vanishes for $x_i=a_i$, $i=1,2,3$, and by Theorem 12.6.2 again, $w_\mu(x_1,x_2,x_3)$ is a proper polynomial identity for $M_2(K)$. Therefore, w'_μ and w''_μ are linearly dependent modulo $B_m\cap T(M_2(K))$ and $k_\mu(M_2(K))\leq 1$. The proof will be completed if we show that $k_\mu(M_2(K))=0$ for $\mu=(1^3)$ and $\mu=(n)$, $n\geq 1$, and if we construct nonzero highest weight vectors in $B_m(M_2(K))$ for all other partitions $\mu=(\mu_1,\mu_2,\mu_3)$. The multiplicity of $W_m(n)$ in $K\langle V_m\rangle$ is 1 and the corresponding highest weight vector is x_1^n. Clearly $x_1^n\notin B_m$ for $n\geq 1$ and, therefore, $k_{(n)}(M_2(K))=0$. Similarly, $k_{(1^3)}(M_2(K))=0$ because the only submodule $W_m(1^3)$ of $K\langle V_m\rangle$ is generated by the standard polynomial $s_3(x_1,x_2,x_3)$ which does not belong to B_m. Now, let $\mu=(\mu_1,\mu_2,\mu_3)$ be another partition. We assume that $\mu\vdash n>1$, since $W_m(\mu)=K$ for $\mu=(0)$. Let

$$w_\mu=s_2(x_1,x_2)(\mathrm{ad}x_1)^{\mu_1-\mu_2}s_3(\mathrm{ad}x_1,\mathrm{ad}x_2,\mathrm{ad}x_3)^{\mu_3}s_2(x_1,x_2)^{\mu_2-\mu_3-1}$$

when $\mu_2\neq\mu_3$ and, for $\mu_2=\mu_3$,

$$w_\mu = \sum_{\sigma \in S_3} (\text{sign}\sigma) x_{\sigma(1)} (\text{ad}x_1)^{\mu_1-\mu_3} s_3(\text{ad}x_1, \text{ad}x_2, \text{ad}x_3)^{\mu_3-1}[x_{\sigma(2)}, x_{\sigma(3)}].$$

The polynomial w_μ has the desired skew-symmetry and all variables are in commutators. Hence w_μ is a highest weight vector of $W_m(\mu) \subset B_m$. Direct verifications show that $w_\mu(b_1, b_2, b_3) \neq 0$ for the matrices b_1, b_2, b_3 in Remark 12.6.3. Therefore, $w_\mu \notin B_m \cap T(M_2(K))$.

Now we are able to give an explicit formula for the cocharacters of the T-ideal of the matrix algebra $M_2(K)$.

Theorem 12.6.5 (Formanek [105], Drensky [74]) *The cocharacter sequence of the T-ideal $T(M_2(K))$ is*

$$\chi_n(M_2(K)) = \sum_{\lambda \vdash n} m_\lambda(M_2(K))\chi_\lambda, n = 0, 1, 2, \ldots,$$

where $\lambda = (\lambda_1, \lambda_2, \lambda_3, \lambda_4)$ and

(i) $m_{(n)}(M_2(K)) = 1$;

(ii) $m_{(\lambda_1,\lambda_2)}(M_2(K)) = (\lambda_1 - \lambda_2 + 1)\lambda_2$, *if* $\lambda_2 > 0$;

(iii) $m_{(\lambda_1,1,1,\lambda_4)}(M_2(K)) = \lambda_1(2 - \lambda_4) - 1$;

(iv) $m_\lambda(M_2(K)) = (\lambda_1 - \lambda_2 + 1)(\lambda_2 - \lambda_3 + 1)(\lambda_3 - \lambda_4 + 1)$ *for all other partitions.*

Proof. We apply Theorem 12.5.4 to the decomposition of $B_m(M_2(K))$ in Theorem 12.6.4, bearing in mind that $B_m^{(n)}(M_2(K))$ and $\Gamma_n(M_2(K))$ have the same module structure (see Theorem 12.4.20 and Remark 12.4.21). Let $\lambda = (\lambda_1, \ldots, \lambda_m)$ be a partition of n. By Theorem 12.5.4, the multiplicity $m_\lambda(M_2(K))$ in the cocharacter sequence of $M_2(K)$ is equal to the sum of all multiplicities $k_\mu(M_2(K))$, $\mu = (\mu_1, \ldots, \mu_m)$, where $\lambda_i \geq \mu_i \geq \lambda_{i+1}$. By Theorem 12.6.4, $k_\mu(M_2(K)) = 1$ for $\mu = (\mu_1, \mu_2, \mu_3)$, $\mu \neq (p)$, $p > 0$, and $\mu \neq (1^3)$ and $k_\mu(M_2(K)) = 0$ in all other cases. Clearly, $m_\lambda(M_2(K)) = 0$ if $\lambda_5 \neq 0$ and we have to handle the case $\lambda = (\lambda_1, \lambda_2, \lambda_3, \lambda_4)$. Let us assume that $k_\mu(M_2(K)) = 1$ for all $\mu = (\mu_1, \mu_2, \mu_3)$. For a fixed λ we obtain that the contribution to $m_\lambda(M_2(K))$ is due to those μ which satisfy

$$\mu_1 = \lambda_2, \lambda_2 + 1, \ldots, \lambda_1; \mu_2 = \lambda_3, \lambda_3 + 1, \ldots, \lambda_2; \mu_3 = \lambda_4, \lambda_4 + 1, \ldots, \lambda_3.$$

Hence we obtain

$$(\lambda_1 - \lambda_2 + 1)(\lambda_2 - \lambda_3 + 1)(\lambda_3 - \lambda_4 + 1)$$

possibilities to choose μ and this gives the multiplicity of $m_\lambda(M_2(K))$ in the general case. Now we have to subtract the contribution of the partitions $\mu = (p)$, $p > 0$, and $\mu = (1^3)$, because for them $k_\mu(M_2(K)) = 0$. For $\mu = (p)$ this concerns $\lambda = (n)$ (n possibilities for $\mu = (p)$, $p = 1, 2, \ldots, n$, to be subtracted) and $\lambda = (\lambda_1, \lambda_2)$ ($\lambda_1 - \lambda_2 + 1$ possibilities for $\mu = (p)$, $p = \lambda_2, \lambda_2 + 1, \ldots, \lambda_1$).

Similarly, $\mu = (1^3)$ concerns $\lambda = (\lambda_1, 1^2)$ and $\lambda = (\lambda_1, 1^3)$, when we subtract 1. Easy calculations complete the proof.

Theorem 12.6.2 together with Theorem 4.3.12 allows to calculate the Hilbert series of $F_m(M_2(K))$ and the codimensions of $M_2(K)$.

Exercise 12.6.6 Prove the theorem of Formanek, Halpin and Li [109] that

$$\mathrm{Hilb}(F_2(M_2(K)), t_1, t_2) = \frac{1}{(1-t_1)(1-t_2)}\left(1 + \frac{t_1t_2}{(1-t_1)(1-t_2)(1-t_1t_2)}\right).$$

Hint. Use that

$$\mathrm{Hilb}(F_2(M_2(K)), t_1, t_2) =$$

$$= \frac{1}{(1-t_1)(1-t_2)}\left(\sum_{p\geq 0}\sum_{q\geq 0} S_{(p+q,q)}(t_1,t_2) - \sum_{p\geq 1} S_{(p)}(t_1,t_2)\right),$$

$$S_{(p+q,q)}(t_1,t_2) = (t_1t_2)^q(t_1^p + t_1^{p-1}t_2 + \ldots + t_1t_2^{p-1} + t_2^p) =$$

$$= (t_1t_2)^q\frac{t_1^{p+1} - t_2^{p+1}}{t_1 - t_2}.$$

Compare this proof with the original proof of Formanek, Halpin and Li in [109].

Remark 12.6.7 Since $\mathrm{dim} M_2(K) = 4$, Exercise 12.5.13 gives that the cocharacter sequence of $M_2(K)$ contains only irreducible characters corresponding to $\lambda = (\lambda_1, \lambda_2, \lambda_3, \lambda_4)$ and the multiplicities of $\chi_\lambda(M_2(K))$ are completely determined by the Hilbert series of $F_4(M_2(K))$. Formanek [105] proved that

$$\mathrm{Hilb}(F_4(M_2(K)), t_1, t_2, t_3, t_4) = \prod_{i=1}^{4}\frac{1}{(1-t_i)^2}\left(\frac{1 - t_1t_2t_3t_4}{\prod_{i<j}(1-t_it_j)} - 1\right) +$$

$$+\prod_{i=1}^{4}\frac{1}{1-t_i}\left(1 - e_3(t_1,t_2,t_3,t_4)\right).$$

Another description of the Hilbert series of $F_m(M_2(K))$ is given in [215] and [74].

Theorem 12.6.8 (i) (Procesi [215]) *The codimension series and the codimension sequence of $M_2(K)$ are the following:*

$$c(M_2(K), t) = \sum_{n\geq 0} c_n(M_2(K))t^n = \frac{1}{t^2}\left(1 - 2t - \sqrt{1-4t}\right) -$$

$$-\frac{t^3}{(1-t)^4}+\frac{1}{1-t}-\frac{1}{1-2t};$$

$$c_n(M_2(K))=\frac{1}{n+2}\binom{2n+2}{n+1}-\binom{n}{3}+1-2^n.$$

(ii) (Regev, see [226]) *The n-th codimension $c_n(M_2(K))$ satisfies the asymptotic equality*

$$c_n(M_2(K))\approx\frac{4^{n+1}}{n\sqrt{\pi n}}.$$

Proof. (i) By Theorems 4.3.12 (ii) and 12.6.4, it is sufficient to compute the proper codimension series

$$\gamma(M_2(K),t)=\sum_{k\geq 0}\gamma_k(M_2(K))t^k=$$

$$=\sum_{k\geq 0}\sum_{\mu\vdash k}\dim M(\mu_1,\mu_2,\mu_3)t^k-\sum_{k\geq 1}\dim M(k)t^k-\dim M(1^3)t^3.$$

In order to calculate the first sum, let us consider a hypothetic PI-algebra R such that

$$B_m(R)=B_m/(B_m\cap T(R))=\sum_{p\geq 0}W_m(p^2).$$

As in the proof of Theorem 12.6.5, we obtain that

$$F_m(R)=\sum_{k\geq 0}\sum_{\mu\vdash k}W_m(\mu_1,\mu_2,\mu_3).$$

In virtue of Theorem 4.3.12 (ii),

$$c(R,t)=\frac{1}{1-t}\gamma\left(R,\frac{t}{1-t}\right)\text{ and }\gamma(R,t)=\sum_{p\geq 0}\dim M(p^2).$$

Applying the hook formula of Theorem 12.2.12 (ii), we obtain

$$\dim M(p^2)=\frac{1}{p+1}\binom{2p}{p}$$

and, as a consequence,

$$\gamma(R,t)=\sum_{p\geq 0}\binom{2p}{p}\frac{t^{2p}}{p+1}=\frac{1}{2t^2}\left(1-\sqrt{1-4t^2}\right).$$

Easy calculations complete the proof, because

$$\sum_{k\geq 1}\dim M(k)t^k+\dim M(1^3)t^3=\frac{1}{1-t}-1+t^3.$$

The assertion for the codimensions is equivalent to that for the codimension series and (ii) follows from the Stirling formula

$$n! \approx \sqrt{2\pi n}\left(\frac{n}{e}\right)^n.$$

Remark 12.6.9 Using his method, Regev (see [226]) determined the asymptotic behaviour of the codimension sequence of $M_k(K)$ for any $k > 1$ and showed that

$$c_n(M_k(K)) \approx (2\pi)^{(1-k)/2}2^{(1-k^2)/2}1!2!\dots(k-1)!k^{(k^2+4)/2}n^{(1-k^2)/2}k^{2n+2}.$$

Exercise 12.6.10 Prove the theorem of Drensky [71] for the cocharacter sequence of the Lie algebra $sl_2(K)$:

$$\chi_n(sl_2(K)) = \sum \chi_\lambda,\ n \geq 1,$$

where the summation runs on all partitions $\lambda = (\lambda_1, \lambda_2, \lambda_3) \vdash n$ such that $\lambda_2 \neq 0$ for $n > 1$ and at least one of the integers $\lambda_1 - \lambda_2$ and $\lambda_2 - \lambda_3$ is odd.

Hint. Use that the homogeneous components of degree $n \geq 2$ of the free Lie algebra L_m are contained in $B_m^{(n)} \subset K\langle V_m\rangle$ and in the proof of Theorems 12.6.2 and 12.6.4 we have worked with the basis $\{a_1, a_2, a_3\}$ of the vector space $sl_2(K)$. Repeating the main steps of the proof of Theorem 12.6.4, show that the nonzero highest weight vectors $w_\lambda(x_1, x_2, x_3)$ can be chosen to belong to L_m if one of the integers $\lambda_1 - \lambda_2$ or $\lambda_2 - \lambda_3$ is odd. Show that $w_\lambda(a_1, a_2, a_3)$ are central elements of $M_2(K)$ if both $\lambda_1 - \lambda_2$ and $\lambda_2 - \lambda_3$ are even and hence $w_\lambda(a_1, a_2, a_3) \notin sl_2(K)$.

Exercise 12.6.11 Calculate the Hilbert series

$$\mathrm{Hilb}(F_2(sl_2(K)), t_1, t_2)$$

(the result is a theorem of Bahturin [20]).

Hint. Apply the previous exercise in the spirit of Exercise 12.6.6.

Exercise 12.6.12 Let $I(M_2(K))$ be the ideal of weak polynomial identities of $M_2(K)$ (see Definition 7.3.6). Prove the theorem of Procesi [215] for the $GL_m(K)$-module isomorphism

$$K\langle V_m\rangle/(K\langle V_m\rangle \cap I(M_2(K))) \cong \sum_\lambda W_m(\lambda_1, \lambda_2, \lambda_3).$$

Hint. Repeat the arguments from Exercise 12.6.10 using the highest weight vectors

$$w_\lambda(x_1,x_2,x_3)=s_3^{\lambda_3}(x_1,x_2,x_3)[x_1,x_2]^{\lambda_2-\lambda_3}x_1^{\lambda_1-\lambda_2}.$$

Remark 12.6.13 Razmyslov [217] showed that the ideal $I(M_2(K))$ of weak polynomial identities for $M_2(K)$ is the smallest ideal of $K\langle X\rangle$ which is closed under substitutions of Lie elements and contains the polynomial $[x_1^2,x_2]$. Drensky and Koshlukov [93] proved that $I(M_2(K))$ is the smallest ideal of $K\langle X\rangle$ which is closed under substitutions of linear combinations of the variables (i.e. is a $GL(\text{span}(X))$-module) and contains $[x_1^2,x_2]$ and $s_4(x_1,x_2,x_3,x_4)$. A similar theorem over an infinite field of positive characteristic was established by Koshlukov [151]. Compare these results with Exercise 7.4.4 and its hint. See also [84] for relations with invariant theory of matrices.

Remark 12.6.14 By the theory of Kemer (see Theorem 8.4.10), the simplest T-prime ideals are the commutator ideal and those of the Grassmann algebra, the 2×2 matrix algebra and the algebra $M_{1,1}$. It is known that $T(M_{1,1})=T(E\otimes E)$. Popov [209] proved that

$$T(E\otimes E)=\langle[[x_1,x_2],[x_3,x_4],x_5],\ [[x_1,x_2]^2,x_1]\rangle^T$$

and determined the S_n-module structure of the proper multilinear polynomial identities of $E\otimes E$:

$$\Gamma_n(E\otimes E)=\sum M(p,2^q,1^r),$$

where the summation runs on all partitions $(p,2^q,1^r)\vdash n$ except the trivial cases $p=n>0$, $q=r=0$ (which corresponds to the linearization of x_1^n and $M(n)$ is not contained in Γ_n) and (1^{2k+1}) (related with the standard polynomial of odd degree which is not proper again).

Exercise 12.6.15 Calculate the cocharacter sequence of the tensor square $E\otimes E$ of the Grassmann algebra E.

Hint. Use the result of Popov in Remark 12.6.14 and apply Theorem 12.5.4 as in the proof of Theorem 12.6.5.

Exercise 12.6.16 Let $E(V_2)$ be the Grassmann algebra of the two-dimensional vector space V_2. Calculate the cocharacters and the codimensions of the algebra $E(V_2)\otimes E(V_2)$.

Hint. This is a partial case of a result of Di Vincenzo and Drensky [64] which describes the proper cocharacters of $E\otimes E(V_l)$ and $E(V_k)\otimes E(V_l)$, $k,l\geq 2$. Show that $E(V_2)\otimes E(V_2)$ satisfies the polynomial identities

$$[x_1,x_2,x_3,x_4]=[x_1,x_2,x_3][x_4,x_5]=[x_1,x_2][x_3,x_4,x_5]=0$$

$$[x_1,x_2][x_3,x_4][x_5,x_6]=0$$

and derive that

$$B_m(E(V_2)\otimes E(V_2))\cong K\oplus W_m(1^2)\oplus W_m(2,1)\oplus W_m(2^2)\oplus W_m(1^4).$$

Another possibility is to use that $E(V_2)\otimes E(V_2)$ satisfies all polynomial identities of $E\otimes E$ and to apply the result of Popov in Remark 12.6.14. One can see that all proper polynomials of degree ≥ 5 vanish on $E(V_2)\otimes E(V_2)$ and check which proper highest weight vectors of degree 4 are identities for $E(V_2)\otimes E(V_2)$. One can show even more, that the T-ideal of $K\langle X\rangle$ generated by the polynomial $[x_1,x_2,x_3,x_4]$ coincides with the intersection of the T-ideals $T(E)$ and $T(E(V_2)\otimes E(V_2))$. See the paper by Volichenko [262] for the description of $\langle[x_1,x_2,x_3,x_4]\rangle^T$.

Exercise 12.6.17 Let $C_k=K[t]/(t^k)$ be the polynomial algebra in one variable modulo the ideal generated by t^k, $k>1$. Let

$$R_k=\begin{pmatrix} C & tC\\ tC & C\end{pmatrix}$$

be the subalgebra of the 2×2 matrix algebra with entries from C, i.e. R_k consists of all matrices such that the "other" diagonal contains only polynomials without constant terms. Calculate the proper cocharacters of the algebra R_k.

Hint. Since R_k satisfies all polynomial identities of $M_2(K)$, we obtain that $B_m(R_k)$ is a homomorphic image of $B_m(M_2(K))$. Repeating the arguments of the proof of Theorem 12.6.4, show that $W_m(\mu_1,\mu_2,\mu_3)\subset B_m(R_k)$ if and only if $\mu_2+\mu_3\leq k$. The algebra R_3 plays a role in noncommutative invariant theory (see [82] and [148]).

Exercise 12.6.18 Let R be the algebra of all matrices

$$\begin{pmatrix} \alpha & \gamma_{12} & \gamma_{13}\\ 0 & \beta & \gamma_{23}\\ 0 & 0 & \alpha\end{pmatrix},\ \alpha,\beta,\gamma_{ij}\in K.$$

Calculate the proper cocharacters of R.

Hint. Working modulo the centre of R, we may assume that the noncentral elements of R are of the form

$$\beta e_{22}+\gamma_{12}e_{12}+\gamma_{23}e_{23}.$$

By Exercise 12.5.14, the proper cocharacters are decomposed into a sum of irreducible S_n-characters χ_λ with $\lambda_4=0$. Use that R satisfies the polynomial identities

$$[x_1,x_2][x_3,x_4][x_5,x_6]=[[x_1,x_2][x_3,x_4],x_5]=0$$

and hence all polynomial identities of $E \otimes E$. Apply Remark 12.6.14 and derive that

$$B_m(R) \subset \sum_{q+r\leq 2} W_m(p, 2^q, 1^r).$$

Show that for $\lambda_2 + \lambda_3 > 2$ the highest weight vectors $w_\lambda \in B_m$ vanish on R because involve more than two times the entries of e_{12} and e_{23}. *Answer*:

$$B_m(R) = K \oplus \sum_{p\geq 2} W_m(p-1, 1) \oplus \sum_{p\geq 4}(W_m(p-2, 2) \oplus W_m(p-2, 1^2)).$$

Exercise 12.6.19 Show that the T-ideal $T(R)$ of a PI-algebra R does not contain any standard polynomial s_n, $n \geq 1$, if and only if $T(R) \subseteq T(E)$.

Hint. Use that if $T(R)$ is not contained in $T(E)$, then

$$K\langle X\rangle/(T(E) + T(R))$$

is a proper homomorphic image of $F(E)$. Using the basis of $P_n(E)$ consisting of multilinear elements of the form

$$x_{i_1} \ldots x_{i_{n-2k}}[x_{j_1}, x_{i_2}] \ldots [x_{i_{2k-1}}, x_{j_{2k}}],\ i_1 < \ldots < i_{n-2k},\ j_1 < \ldots < j_{2k},$$

(see Theorem 4.3.11 (ii) and the proof of Theorem 5.1.2) show that every proper subvariety of the variety generated by the Grassmann algebra satisfies some standard identity. Hence $(P_n \cap T(R))/(P_n \cap T(R) \cap T(E))$ contains $M(1^n)$ for some n. Use that the multiplicity of $M(1^n)$ in P_n is equal to 1 and derive that $s_n \in T(R)$.

Exercise 12.6.20 Prove for unitary algebras Theorem 8.2.1 of Kemer [139] describing the PI-algebras with polynomial growth of the codimension sequence. If the algebra R satisfies polynomial identities $f_1 = f_2 = 0$ such that $f_1 \notin T(E)$, $f_2 \notin T(U_2(K))$, then the growth of $c_n(R)$ is polynomial. (The inverse part of the theorem was handled in Exercise 8.2.2.)

Hint. Let the unitary algebra R satisfy polynomial identities $f_1 = f_2 = 0$ such that $f_1 \notin T(E)$, $f_2 \notin T(U_2(K))$. By Exercise 12.6.19, R satisfies some standard identity and by Theorem 8.4.5 of Kemer, some Capelli identity. Hence (Exercises 12.5.12 and 12.4.16) $T(R) = T(F_m(R))$ for some m. The algebra $F_m(R)$ is finitely generated and satisfies the nonmatrix polynomial identity $f_2 = 0$. By the theorem of Latyshev in Remark 5.2.2, $F_m(R)$ satisfies $[x_1, x_2] \ldots [x_{2k-1}, x_{2k}] = 0$ for some k. Using arguments as in the hint to Exercise 12.6.19, $K\langle X\rangle/(T(R) + T(U_2(K)))$ satisfies the Engel identity $x_2\mathrm{ad}^p x_1 = 0$ for some p. Derive from here that R itself satisfies the Engel identity, because the multiplicity of $W_m(p, 1)$ in B_m is equal to 1. Obtain that the Engel identity together with $[x_1, x_2] \ldots [x_{2k-1}, x_{2k}] = 0$ implies that

$B_m^{(n)}(R) = 0$ for n large enough. (The number of the commutators in the products of commutators in $B_m^{(n)}(R)$ is bounded by $k-1$. The length of each commutator $x_j \prod_{i=1}^{m} \operatorname{ad}^{q_i} x_i$ is bounded by $1+(p-1)m$, otherwise it follows from the Engel identity). Since $B_m^{(n)}(R) = 0$ for big n's, then Theorem 4.3.12 (ii) gives the polynomial growth of the codimensions.

Test

As we have mentioned in the preface, the book is based on a graduate course given at the University of Hong Kong. We give the final test for the course. The students had one week to work on 12 problems. Any 6 problems solved were sufficient for "A". Since the level and the purposes of the students were different and not all of them were graduate students working on algebra, the level of difficulty of the 12 problems is also different.

1. Let $\mathbb{H}$ be the $\mathbb{R}$-algebra with basis $\{1, i, j, k\}$ and multiplication

$$i^2 = j^2 = k^2 = -1,\ ij = -ji = k,\ jk = -kj = i,\ ki = -ik = j.$$

(i) Show that $\mathbb{H}$ is an associative algebra.

(ii) Calculate $(\alpha + \beta i + \gamma j + \delta k)(\alpha - \beta i - \gamma j - \delta k)$, $\alpha, \beta, \gamma, \delta \in \mathbb{R}$.

(iii) Show that every nonzero element of $\mathbb{H}$ is invertible, i.e. $\mathbb{H}$ is a noncommutative "field".

2. Let $a_1 = e_{11} - e_{22}$, $a_2 = e_{12} + e_{21}$, $a_3 = e_{12} - e_{21}$ be elements of the algebra of 2×2 matrices with entries from a field K. Let $a_{i_1} \ldots a_{i_n}$ be a product which contains n_1 times a_1, n_2 times a_2, n_3 times a_3, $n_1 + n_2 + n_3 = n$. Let $\varepsilon_i = 0, 1$, $\varepsilon_i \equiv n_i \pmod 2$, $i = 1, 2, 3$. Show that

$$a_{i_1} \ldots a_{i_n} = \alpha a_1^{\varepsilon_1} a_2^{\varepsilon_2} a_3^{\varepsilon_3},\ \alpha \in K.$$

3. Show that the three-dimensional Lie algebra $\mathbb{C}^3$ with basis $\{\mathbf{i}, \mathbf{j}, \mathbf{k}\}$ with the usual vector multiplication

$$(\alpha_1 \mathbf{i} + \beta_1 \mathbf{j} + \gamma_1 \mathbf{k}) \times (\alpha_2 \mathbf{i} + \beta_2 \mathbf{j} + \gamma_2 \mathbf{k}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \alpha_1 & \beta_1 & \gamma_1 \\ \alpha_2 & \beta_2 & \gamma_2 \end{vmatrix}$$

is isomorphic to the Lie algebra $sl_2(\mathbb{C})$ of all traceless 2×2 matrices with entries from $\mathbb{C}$.

4. Let G be the two-dimensional Lie algebra with basis $\{a, b\}$ and multiplication $[a, b] = a$. Present in a canonical form (as in the Poincaré-Birkhoff-Witt theorem) the element $f = baabab$ of the universal enveloping algebra $U(G)$.

5. Calculate the codimensions of the polynomial identities of the algebra of 2×2 upper triangular matrices over a field of characteristic 0.

6. Show that over a field of characteristic 0, the polynomial identity

$$[[[x, y], [x, z]], x] = 0$$

is a consequence of the standard identity $s_4(x_1, x_2, x_3, x_4) = 0$.

7. Let $C_k = K[t]/(t^k)$ be the polynomial algebra in one variable modulo the ideal generated by t^k, $k > 1$. Let

$$R_k = \begin{pmatrix} C & tC \\ tC & C \end{pmatrix}$$

be the subalgebra of the 2×2 matrix algebra with entries from C, i.e. R_k consists of all matrices such that the "other" diagonal contains only polynomials without constant terms. Show that R_k satisfies the polynomial identity

$$[x_1, x_2] \ldots [x_{2k-1}, x_{2k}] = 0.$$

8. Let $\text{char} K = 0$ and let $k \in \mathbb{N}$.

(i) Show that in the free associative algebra without 1 the elements yx^k, $x^k z$ and $yx^k z$ are linear combinations of some $u_1^k, \ldots, u_n^k$.

(ii) Show that in the free nonunitary associative algebra $K\langle X\rangle$ the T-ideal generated by x^k coincides with the vector space spanned by all u^k, $u \in K\langle X\rangle$.

9. Let $a_0, a_1, a_2, \ldots$ be the sequence defined by

$$a_0 = 0,\ a_2 = 1,\ a_2 = 2,\ a_{n+3} = 3a_{n+1} - 2a_n.$$

Find a closed formula for a_n (i.e. a formula which expresses a_n in terms of n only and does not involve the previous elements of the sequence).

10. Let U_k be the algebra of $k \times k$ upper triangular matrices with entries from an infinite field. Calculate the Gelfand-Kirillov dimension of the relatively free algebra

$$F_m(U_k) = K\langle x_1, \ldots, x_m\rangle/(T(U_k) \cap K\langle x_1, \ldots, x_m\rangle),\ m > 1.$$

11. Let

$$\nu = (x - 2y(y^2 + xz) - z(y^2 + xz)^2,\ y + z(y^2 + xz),\ z, t)$$

be the Nagata automorphism acting on the polynomial algebra $K[x, y, z, t]$. Decompose ν as a product of linear and triangular automorphisms. Write explicitly the action of these automorphisms on the generators x, y, z, t.

12. Show that the following endomorphism of $K\langle x,y,z\rangle$ is an automorphism and find its inverse

$$\phi=(x+y(xy-yz),\ y,\ z+(xy-yz)y).$$

Hints

1. The algebra $\mathbb{H}$ is the *quaternion algebra* (compare with the quaternion group in Exercise 12.1.19).

(i) Verify the associative law on the basis vectors i,j,k.

(ii) Answer:

$$(\alpha+\beta i+\gamma j+\delta k)(\alpha-\beta i-\gamma j-\delta k)=\alpha^2+\beta^2+\gamma^2+\delta^2.$$

(iii) Answer:

$$(\alpha+\beta i+\gamma j+\delta k)^{-1}=\frac{\alpha-\beta i-\gamma j-\delta k}{\alpha^2+\beta^2+\gamma^2+\delta^2},\ (\alpha,\beta,\gamma,\delta)\neq(0,0,0,0).$$

2. This is Exercise 12.6.1. (Chapter 12 was not included in the originally given course.)

3. Define the isomorphism $\phi:\mathbb{C}^3\to sl_2(\mathbb{C})$ by

$$\phi(\mathbf{i})=\alpha a_1,\ \phi(\mathbf{j})=\beta a_2,\ \phi(\mathbf{k})=\gamma a_3,$$

where a_1,a_2,a_3 are as in Problem 2 and $\alpha,\beta,\gamma\in\mathbb{C}$ are chosen in such a way that

$$\phi(\mathbf{k})=\phi(\mathbf{i}\times\mathbf{j})=[\phi(\mathbf{i}),\phi(\mathbf{j})]=2\alpha\beta a_1a_2=2\gamma a_3,$$

similarly for $\mathbf{i}=\mathbf{j}\times\mathbf{k}$ and $\mathbf{j}=\mathbf{k}\times\mathbf{i}$. Use Remark 12.6.3.

4. Use that in $U(G)$

$$[a,b]=a,\ ba=ab-a$$

and express f as $\sum\alpha_{kl}a^kb^l$.

5. This is a partial case of Exercise 5.2.7. Use that the exponential codimension series is

$$\tilde{c}(U_2(K),t)=\frac{1}{1-t}(1-((t-1)e^t+1)^2).$$

Another approach is to use directly that $\gamma_n(U_2(K))=n-1$, $n>0$, and

$$c_n(U_2(K))=\sum_{p=0}^{n}\binom{n}{p}\gamma_p(U_2(K)).$$

Answer: $c_n(U_2(K)) = 2 + 2^{n-1}(n-2)$, $n \geq 1$.

6. Calculate $s_4(x_1, x_2, x_3, x_1^2)$.

7. Show that $[R_k, R_k] \subseteq tR_k$. Compare this problem with Exercise 12.6.17.

8. (i) The partial linearizations of a polynomial identity $f(x_1, \ldots, x_m)$ are obtained by Vandermonde arguments and, therefore, are linear combinations of $f(u_1, \ldots, u_m)$, $u_i \in K\langle X\rangle$ $(1 \notin K\langle X\rangle)$. Hence the partial linearization

$$f_k(x, y) = x^{k-1}y + x^{k-2}yx + \ldots + xyx^{k-2} + yx^{k-1}$$

is a linear combination of u_i^k, $u_i \in K\langle X\rangle$. We consider $f_k(x + x^2, y)$, take the homogeneous component of degree $k+1$ (Vandermonde arguments again) and obtain

$$(x^k y + yx^k) + (k-2)(x^k y + x^{k-1}yx + \ldots + xyx^{k-1} + yx^k) =$$
$$= (x^k y + yx^k) + (k-2)f_k(x, y).$$

Since kx^{k+1} is the homogeneous component of degree $k+1$ of $(x + x^2)^k$, we obtain that $f_{k+1}(x, y)$ is also a linear combination of some u_i^k. Finally, since $\mathrm{ad}y$ is a derivation, we see that

$$x^k y - yx^k = [x^k, y] = x^k \mathrm{ad}y = f_k(x, [x, y]),$$

i.e. $x^k y + yx^k$ and $x^k y - yx^k$ are both linear combinations of u_i^k and the same holds for yx^k (similarly for $x^k z$). Now

$$yx^k z = (yx^k)z = \sum \alpha_i u_i^k z = \sum \alpha_i \sum \beta_{ij} v_{ij}^k.$$

(ii) This is a precised version of Nagata-Higman Theorem 8.3.2 (see the proof of Higman [127]). Since

$$\langle x^k\rangle^T = \left\{\sum u_i v_i^k w_i \mid u_i, v_i, w_i \in K\langle X\rangle\right\},$$

and $yx^k z = \sum \alpha_j g_j^k(x, y, z)$, $g_j \in K\langle X\rangle$, we obtain that

$$u_i v_i^k w_i = \sum \alpha_i g_j^k(v_i, u_i, w_i),$$

and this completes the proof.

9. Use Exercise 6.1.3 or prove directly by induction that $a_n = n$.

10. This is Exercise 9.4.5.

11. Follow the proofs of Theorem 10.4.8 and Corollary 10.4.9.

12. Show that $\phi(xy - yz) = xy - yz$. Answer:

$$\phi^{-1} = (x - y(xy - yz), y, z - (xy - yz)y).$$

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